SIMILARITY AND ERGODIC THEORY OF POSITIVE LINEAR MAPS

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Abstract. In this paper we study the operator inequality $\varphi(X) \leq X$ and the operator equation $\varphi(X) = X$, where φ is a w^*-continuous positive (resp. completely positive) linear map on $B(\mathcal{H})$. We show that their solutions are in one-to-one correspondence with a class of Poisson transforms on Cuntz-Toeplitz C^*-algebras, if φ is completely positive. Canonical decompositions, ergodic type theorems, and lifting theorems are obtained and used to provide a complete description of all solutions, when $\varphi(I) \leq I$.

We show that the above-mentioned inequality (resp. equation) and the structure of its solutions have strong implications in connection with representations of Cuntz-Toeplitz C^*-algebras, common invariant subspaces for n-tuples of operators, similarity of positive linear maps, and numerical invariants associated with Hilbert modules over $\mathbb{C}\mathbb{F}_n^+$, the complex free semigroup algebra generated by the free semigroup on n generators.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Given a positive linear map $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$, we define the following sets:

(i) $C_{\leq}(\varphi)^+ := \{X \in B(\mathcal{H}) : X \geq 0 \text{ and } \varphi(X) \leq X\}$ (noncommutative cone);
(ii) $C_{=}(\varphi) := \{X \in B(\mathcal{H}) : \varphi(X) = X\}$ (fixed-point operator space).

We will refer to these sets as the $C(\varphi)$-sets associated with φ. The structure of these sets plays a distinguished role in the ergodic theory of positive maps [21], the classification of the endomorphisms of $B(\mathcal{H})$ (eg. [41], [42], [5], [6], [25], [11], [12], [13]), and the representation theory of Cuntz algebras (eg. [17], [30], [25], [11], [12], [13], [18], [19]). In the particular case when $T \in B(\mathcal{H})$, $\|T\| \leq 1$, and $\varphi_T(X) := TXT^*$ these sets were studied by R.G. Douglas in [20] and by Sz.-Nagy and Foiaş [45] in connection with T-Toeplitz operators. The operator space $C_{=}(\varphi_T)$ was also studied in [15] and [16].

In this paper we study the structure of the $C(\varphi)$-sets associated with a w^*-continuous positive (resp. completely positive) linear map φ on $B(\mathcal{H})$ and its connections with Poisson transforms on Cuntz-Toeplitz C^*-algebras, common invariant subspaces for n-tuples of operators, similarity of positive linear maps, and numerical invariants associated with Hilbert modules over $\mathbb{C}\mathbb{F}_n^+$, the complex free semigroup algebra generated by the free semigroup on n generators.

It is well-known (see eg. [22]) that any w^*-continuous completely positive linear map φ on $B(\mathcal{H})$ is determined by a sequence $\{A_i\}_{i=1}^n$ ($n \in \mathbb{N}$ or $n = \infty$) of bounded operators on $\mathcal{H}$, in

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the sense that

$$\varphi(X) := \sum_{i=1}^{n} A_i X A_i^*, \quad X \in B(\mathcal{H}), \tag{1.1}$$

where, if $n = \infty$, the convergence is in the w^*-topology. In Section 2, we show that the positive solutions of the operator inequality $\varphi(X) \leq X$ are intimately related to a class of Poisson tranforms on $C^*(S_1, \ldots, S_n)$, the Cuntz-Toeplitz C^*-algebra generated by the left creation operators $S_1, \ldots, S_n$ on the full Fock space. More precisely, we prove that an operator D is in $C_{\leq}(\varphi)^+$ if and only if there is a Poisson transform

$$P_{\varphi,D} : C^*(S_1, \ldots, S_n) \to B(\mathcal{H})$$

with the following properties:

(i) $P_{\varphi,D}$ is a completely positive linear map;
(ii) $\|P_{\varphi,D}\|_{cb} \leq \|D\|$;
(iii) $P_{\varphi,D}(I) = D$ and

$$P_{\varphi,D}(S_\alpha S_\beta^*) = A_\alpha D A_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

When $A_i A_j = A_j A_i$, $i, j = 1, \ldots, n$, the result remains true if we replace the left creation operators $S_1, \ldots, S_n$ by their compressions $B_1, \ldots, B_n$ to the symmetric Fock space.

Let us mention that, in the particular case when $\varphi(I) \leq I$ and $D := I$, the Poisson transform associated with (φ, I) was introduced and studied in [37] in connection with a noncommutative von Neumann inequality for row contractions [33]. Several applications of these Poisson transforms were considered in [37], [7], [39], [2], [3], [38], and recently in [9], [10], [40], and [1]. We refer to [4], [27], and [28] for results on completely bounded maps and operator spaces.

In Section 3, we present canonical decompositions (see Theorem 3.1), ergodic type results (see Theorem 3.2), and lifting theorems (see Theorem 3.5) for w^*-continuous positive linear maps on $B(\mathcal{H})$. These results together with those from Section 2 are used to provide a complete description of the $C(\varphi)$-sets (see Theorem 3.8 and Corollary 3.9), when φ is a w^*-continuous completely positive linear map with $\varphi(I) \leq I$. When we drop the condition $\varphi(I) \leq I$, we also obtain characterizarions of the $C(\varphi)$-sets (see Theorem 3.4 and Theorem 3.7). An important role in this investigation is played by the noncommutative dilation theory for sequences of operators [23], [14], [29], [30], [31], and [34] (see [44] for the classical dilation theory). For related results when $\varphi(I) = I$ we mention [13].

In Section 4, we show that there is a strong connection between the positive solutions of the operator inequality $\varphi(X) \leq X$, where φ is a w^*-continuous completely positive linear map on $B(\mathcal{H})$, defined as in (1.1), and the common invariant subspaces for the n-tuple of operators $\{A_i\}_{i=1}^n$. In this direction, we obtain invariant subspace theorems (eg. Theorem 4.3) and Wold type decomposition theorems for w^*-continuous completely positive linear maps on $B(\mathcal{H})$ (eg. Theorem 4.7). The latter results generalize the classical Wold decomposition for isometries, as well as the one obtained in [30] for isometries with orthogonal ranges.

Section 5 is devoted to similarity of positive linear maps on $B(\mathcal{H})$. We say that two linear maps $\varphi, \lambda : B(\mathcal{H}) \to B(\mathcal{H})$ are similar if there is an invertible operator $R \in B(\mathcal{H})$ such that

$$\varphi(RXR^*) = R\lambda(X)R^*, \quad \text{for any } X \in B(\mathcal{H}).$$

Notice that this relation is equivalent to

$$\varphi = \psi_R \circ \lambda \circ \psi_R^{-1},$$

where $\psi_R(X) := RXR^*$, $X \in B(\mathcal{H})$. This shows that the discrete semigroups of completely positive maps $\{\varphi^k\}_{k=0}^\infty$ and $\{\lambda^k\}_{k=0}^\infty$ are also similar. Moreover, $D \in C_{\leq}(\lambda)^+$ if and only if $RDR^* \in C_{\leq}(\varphi)^+$. In this section we provide necessary and sufficient conditions for a w^*-continuous positive linear map φ on $B(\mathcal{H})$ to be similar to a positive linear map λ on $B(\mathcal{H})$ satisfying one of the following properties:

(i) $\lambda(I) = I$ (see Theorem 5.1);
(ii) $\|\lambda\| < 1$ (see Theorem 5.9);
(iii) λ is a pure completely positive linear map with $\|\lambda\| \leq 1$ (see Theorem 5.11);
(iv) λ is a completely positive linear map with $\|\lambda\| \leq 1$ (see Theorem 5.13).

We show that these similarities are strongly related to the existence of invertible positive solutions of the operator inequality $\varphi(X) \leq X$ or equation $\varphi(X) = X$.

In [8], Arveson introduced a notion of curvature and Euler characteristic for finite rank contractive Hilbert modules over $\mathbb{C}[z_1, \ldots, z_n]$, the complex unital algebra of all polynomials in n commuting variables. Noncommutative analogues of these notions were introduced and studied by the author in [39] and, independently, by D. Kribs [24]. The Poisson transforms of Section 2 are used in Section 6 to define certain numerical invariants associated with (not necessarily contractive) Hilbert modules over the free semigroup algebra $\mathbb{C}\mathbb{F}_n^+$. We extend and refine some of the results from [39]. Any Hilbert module $\mathcal{H}$ over $\mathbb{C}\mathbb{F}_n^+$ corresponds to a unique w^*-continuous completely positive map φ on $B(\mathcal{H})$ and therefore to a unique noncommutative cone $C_{\leq}(\varphi)^+$. A notion of $*$-curvature $\operatorname{curv}_*(\varphi, D)$ and Euler characteristic $\chi(\varphi, D)$ are associated with each ordered pair (φ, D), where $D \in C_{\leq}(\varphi)^+$. In this section, we obtain asymptotic formulas and basic properties for both the $*$-curvature and the Euler characteristic associated with (φ, D). In the particular case when $\mathcal{H}$ is a contractive Hilbert modules over $\mathbb{C}\mathbb{F}_n^+$ and $D := I$, our two variable invariant

$$F(\varphi, I) := (\|\varphi^*(I)\|, \ \operatorname{curv}_*(\varphi, I))$$

is a refinement of the curvature invariant from [39] and [24].

2. Poisson transforms associated with completely positive maps

A Poisson transform on the Cuntz-Toeplitz algebra $C^*(S_1, \ldots, S_n)$ is associated with each pair (φ, D), where φ is a w^*-continuous completely positive linear map on $B(\mathcal{H})$ and $D \in B(\mathcal{H})$ is a positive operator such that $\varphi(D) \leq D$. The main result of this section (see Theorem 2.1) shows that the elements of the noncommutative cone $C_{\leq}(\varphi)^+$ are in one-to-one correspondence with the elements of a class of Poisson transforms on Cuntz-Toeplitz algebras. On the other hand, we prove that there is a strong connection between the fixed-point operator space $C_{=}(\varphi)$ and a class of Poisson transforms on the Cuntz algebra $\mathcal{O}_n$.

Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$, where $n \in \{1, 2, \ldots\}$ or $n = \infty$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k \geq 0} H_n^{\otimes k},$$

where $H_n^{\otimes 0} := \mathbb{C}1$ and $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. Define the left creation operators $S_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \ldots, n$, by

$$S_i f := e_i \otimes f, \ f \in F^2(H_n).$$

The noncommutative analytic Toeplitz algebra F_n^∞ is the WOT-closed algebra generated by the left creation operators $S_1, \ldots, S_n$ and the identity. This algebra and its norm-closed version (the noncommutative disc algebra $\mathcal{A}_n$) were introduced by the author in [33] in connection with a noncommutative von Neumann inequality.

Let $\mathbb{F}_n^+$ be the free semigroup with n generators $g_1, \ldots, g_n$ and neutral element g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := k$, if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, and $|\alpha| := 0$, if $\alpha = g_0$. We also define $e_\alpha := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ and $e_{g_0} := 1$. It is clear that $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis of $F^2(H_n)$. If $T_1, \ldots, T_n \in B(\mathcal{H})$, define $T_\alpha := T_{i_1} T_{i_2} \cdots T_{i_k}$, if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$ and $T_{g_0} := I$, the identity on $\mathcal{H}$.

Let $\mathcal{H}$ be a separable Hilbert space. Any w^*-continuous completely positive linear map φ on $B(\mathcal{H})$ is determined by a sequence $\{A_i\}_{i=1}^n$ ($n \in \mathbb{N}$ or $n = \infty$) of bounded operators on $\mathcal{H}$, in the sense that

$$\varphi(X) := \sum_{i=1}^n A_i X A_i^*, \quad X \in B(\mathcal{H}),$$

where, if $n = \infty$, the convergence is in the w^*-topology. Fix such a map φ and let $D \in B(\mathcal{H})$ be a positive operator such that $\varphi(D) \leq D$. Denote $\varphi_r := r^2 \varphi$, $0 < r \leq 1$, and define the defect operator $\Delta_r := [D - \varphi_r(D)]^{1/2}$. Notice that, if $0 < r < 1$, then

$$\begin{aligned}\sum_{k=0}^\infty \varphi_r^k(\Delta_r^2) &= D - \varphi_r(D) + \varphi_r(D - \varphi_r(D)) + \cdots \\ &= D - \lim_{n\to\infty} r^{2n} \varphi^n(D) = D.\end{aligned}$$

If $r = 1$, then $\sum_{k=0}^\infty \varphi_r^k(\Delta_r^2) = D - \varphi^\infty(D)$, where $\varphi^\infty(D) := \text{SOT} - \lim\limits_{k\to\infty} \varphi^k(D)$ exists since the sequence of positive operators $\{\varphi^k(D)\}_{k=0}^\infty$ is decreasing.

We introduce the Poisson kernel associated with the ordered pair (φ, D) as the family of operators $K_{\varphi,D,r} : \mathcal{H} \to F^2(H_n) \otimes \mathcal{H}$, $0 < r \leq 1$, defined by

$$K_{\varphi,D,r} h := \sum_{k=0}^\infty \sum_{\alpha \in \mathbb{F}_n^+, |\alpha| = k} e_\alpha \otimes r^{|\alpha|} \Delta_r A_\alpha^* h, \quad h \in \mathcal{H}. \tag{2.1}$$

When $r = 1$, we denote $K_{\varphi,D} := K_{\varphi,D,1}$. Notice that, if $0 < r < 1$, then

$$K_{\varphi,D,r}^* K_{\varphi,D,r} = \sum_{k=0}^\infty \varphi_r^k(\Delta_r^2) = D. \tag{2.2}$$

When $r = 1$, we have

$$K_{\varphi,D}^* K_{\varphi,D} = D - \varphi^\infty(D). \tag{2.3}$$

Due to relation (2.1), for any $i = 1, \ldots, n$, and $0 < r \leq 1$, we have

$$K_{\varphi,D,r}(r A_i^*) = (S_i^* \otimes I) K_{\varphi,D,r}. \tag{2.4}$$

Let $C^*(S_1, \ldots, S_n)$ be the C^*-algebra generated by $S_1, \ldots, S_n$. For $0 < r \leq 1$, define the operator $P_{\varphi,D,r} : C^*(S_1, \ldots, S_n) \to B(\mathcal{H})$ by setting

$$P_{\varphi,D,r}(f) := K_{\varphi,D,r}^* (f \otimes I) K_{\varphi,D,r}, \quad f \in C^*(S_1, \ldots, S_n). \tag{2.5}$$

Using relation (2.4) when $0 < r < 1$, we have

$$K_{\varphi,D,r}^* (S_\alpha S_\beta^* \otimes I) K_{\varphi,D,r} = r^{|\alpha| + |\beta|} A_\alpha D A_\beta, \quad \alpha, \beta \in \mathbb{F}_n^+. \tag{2.6}$$

Hence, and using relations (2.2) and (2.5), we infer that $P_{\varphi,D,r}$ is a completely positive linear map and

$$\|P_{\varphi,D,r}\|_{cb} \leq \|D\|, \quad \text{for any } 0 < r < 1. \tag{2.7}$$

Now we can prove the main result of this section, which shows that the elements of the noncommutative cone $C_{\leq}(\varphi)^+$ are in one-to-one correspondence with the elements of a class of Poisson transforms on Cuntz-Toeplitz algebras.

Theorem 2.1. *Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ defined by*

$$\varphi(X) := \sum_{i=1}^{n} A_i X A_i^*, \quad X \in B(\mathcal{H}),$$

and let $D \in B(\mathcal{H})$ be a positive operator such that $\varphi(D) \leq D$. Then the Poisson transform

$$P_{\varphi,D} : C^*(S_1, \ldots, S_n) \to B(\mathcal{H}), \quad P_{\varphi,D}(f) := \lim_{r \to 1} K^*_{\varphi,D,r}(f \otimes I) K_{\varphi,D,r},$$

where the limit exists in the uniform norm, has the following properties:

- (i) *$P_{\varphi,D}$ is a completely positive linear map;*
- (ii) *$\|P_{\varphi,D}\|_{cb} \leq \|D\|$;*
- (iii) *$P_{\varphi,D}(I) = D$ and*

$$P_{\varphi,D}(S_\alpha S_\beta^*) = A_\alpha D A_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

Proof. If $q(S_1, \ldots, S_n) := \sum_{\alpha,\beta \in \mathbb{F}_n^+} a_{\alpha\beta} S_\alpha S_\beta^*$ is a polynomial in $C^*(S_1, \ldots, S_n)$ define

$$q^D(A_1, \ldots, A_n) := \sum_{\alpha,\beta \in \mathbb{F}_n^+} a_{\alpha\beta} A_\alpha D A_\beta^*.$$

The definition is correct since, according to (2.2) and (2.6), we have

$$\|q^D(A_1, \ldots, A_n)\| \leq \|D\| \|q(S_1, \ldots, S_n)\|. \tag{2.8}$$

Now, if $f \in C^*(S_1, \ldots, S_n)$ and $q_k(S_1, \ldots, S_n)$ is an arbitrary sequence of polynomials in $C^*(S_1, \ldots, S_n)$ convergent to f, we define the operator

$$f^D(A_1, \ldots, A_n) := \lim_{k \to \infty} q_k^D(A_1, \ldots, A_n). \tag{2.9}$$

Taking into account relation (2.8), it is clear that the operator f^D is well-defined and

$$\|f^D(A_1, \ldots, A_n)\| \leq \|D\| \|f\|.$$

According to relations (2.6) and (2.7), we have

$$\|q_k^D(rA_1, \ldots, rA_n)\| \leq \|D\| \|q_k(S_1, \ldots, S_n)\|,$$

for any $0 < r \leq 1$. Since $P_{\varphi,D,r}$ is a bounded linear operator, we have

$$\begin{aligned} f^D(rA_1, \ldots, rA_n) &:= \lim_{k \to \infty} q_k^D(rA_1, \ldots, rA_n) \\ &= \lim_{k \to \infty} P_{\varphi,D,r}(q_k(S_1, \ldots, S_n)) = P_{\varphi,D,r}(f), \end{aligned} \tag{2.10}$$

for any $0 < r < 1$. Using relations (2.9), (2.10), the fact that $\|f - q_k\| \to 0$ as $k \to \infty$, and

$$\lim_{r \to 1} q_k^D(rA_1, \ldots, rA_n) = q_k^D(A_1, \ldots, A_n),$$

we can easily prove that

$$\lim_{r\to 1} P_{\varphi,D,r}(f) = f^D(A_1,\dots,A_n)$$

in the uniform norm. For any $0<r<1$, $P_{\varphi,D,r}$ is a completely positive linear map. Hence, and using relations (2.6), (2.7), we infer that $P_{\varphi,D}$ is a completely positive map with $\|P_{\varphi,D}\|_{cb}\leq \|D\|$. The condition (iii) is clearly satisfied due to relations (2.2) and (2.6). The proof is complete. □

Let us remark that if φ is a pure completely positive map, i.e., $\varphi^k(I)\to 0$ strongly, as $k\to\infty$, then the Poisson transform $P_{\varphi,D}$ satisfies the equation

$$P_{\varphi,D} f = K^*_{\varphi,D}(f\otimes I)K_{\varphi,D},\quad f\in C^*(S_1,\dots,S_n).$$

We should also mention that Theorem 2.1 actually shows that given a w^*-continuous completely positive linear map φ on $B(\mathcal{H})$, there exists $D\geq 0$ such that $\varphi(D)\leq D$ if and only if there is a Poisson transform $P_{\varphi,D}$ with the properties (i), (ii), and (iii) from Theorem 2.1. It remains to prove one implication. Indeed, if we assume that $P_{\varphi,D}$ satisfies the above-mentioned conditions, then

$$\varphi(D) = \sum_{i=1}^n A_i D A_i^* = P_{\varphi,D}\left(\sum_{i=1}^n S_i S_i^*\right) \leq P_{\varphi,D}(I) = D.$$

Corollary 2.2. *If $\varphi(X) := \sum_{i=1}^n A_i X A_i^*$ ($n\in\mathbb{N}$ or $n=\infty$) is a w^*-continuous completely positive linear map with $A_iA_j = A_jA_i$, $i,j=1,\dots,n$, then Theorem 2.1 remains true if we replace the left creation operators $\{S_1,\dots,S_n\}$ by their compressions $\{B_1,\dots,B_n\}$ to the symmetric Fock space $F_s^2(H_n)$.*

Proof. Since $F_s^2(H_n)\subset F^2(H_n)$ is an invariant subspace under each S_i^*, $i=1,\dots,n$, we have

$$P_{F_s^2(H_n)} S_\alpha S_\beta^* | F_s^2(H_n) = B_\alpha B_\beta^*,\quad \alpha,\beta\in\mathbb{F}_n^+.$$

On the other hand, since the operators A_i are commuting, the Poisson kernel $K_{\varphi,D,r}$ takes values in $F_s^2(H_n)\otimes\mathcal{H}$ for any $0<r<1$. Hence, and using relation (2.6), we deduce that

$$\begin{aligned} K^*_{\varphi,D,r}(B_\alpha B_\beta^*\otimes I)K_{\varphi,D,r} &= K^*_{\varphi,D,r}(S_\alpha S_\beta^*\otimes I)K_{\varphi,D,r}\\ &= r^{|\alpha|+|\beta|}A_\alpha D A_\beta,\quad \alpha,\beta\in\mathbb{F}_n^+. \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1. □

We recall [17] that if $n\geq 2$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by elements $v_1,\dots,v_n$ subject to the relations

$$v_i^* v_j = \delta_{ij} I \quad\text{and}\quad \sum_{i=1}^n v_i v_i^* = I.$$

The following result shows that there is a strong connection between the fixed-point operator space $C_=(\varphi)$ and a class of Poisson transforms on the Cuntz algebra $\mathcal{O}_n$. The proof is based on noncommutative dilation theory [30].

Theorem 2.3. *If $\varphi(X) := \sum_{i=1}^n A_i X A_i^*$ ($n\geq 2$ or $n=\infty$) is a w^*-continuous completely positive linear map on $B(\mathcal{H})$ and $D\in B(\mathcal{H})$ is an invertible positive solution of the equation $\varphi(X)=X$, then there is a unique completely positive linear map $\Phi_{\varphi,D}:\mathcal{O}_n\to B(\mathcal{H})$ such that $\Phi_{\varphi,D}(I)=D$ and*

$$\Phi_{\varphi,D}(v_\alpha v_\beta^*) = A_\alpha D A_\beta^*,\quad \alpha,\beta\in\mathbb{F}_n^+,$$

where $\{v_1, \ldots, v_n\}$ is a system of generators for the Cuntz algebra $\mathcal{O}_n$. Moreover, if $\varphi(I) = I$ and D is a positive operator such that $\varphi(D) = D$, then the result remains true.

Proof. Assume D is an invertible positive operator with $\varphi(D) = D$ and set $T_i := D^{-1/2} A_i D^{1/2}$, $i = 1, \ldots, n$. Notice that

$$\sum_{i=1}^n T_i T_i^* = D^{-1/2} \varphi(D) D^{1/2} = I_{\mathcal{H}}.$$

According to [30], the minimal isometric dilation of $[T_1, \ldots, T_n]$ is $[V_1, \ldots, V_n]$, where V_i are isometries on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that

$$\sum_{i=1}^n V_i V_i^* = I_{\mathcal{K}}, \ V_i^*|\mathcal{H} = T_i^*, \text{ and } \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H} = \mathcal{K}.$$

Therefore, there is a unique unital completely contractive linear map $\Psi : C^*(V_1, \ldots, V_n) \to B(\mathcal{H})$ such that $\Psi(V_\alpha V_\beta^*) = T_\alpha T_\beta^*$, $\alpha, \beta \in \mathbb{F}_n^+$. Hence, we infer that $\Psi_{\varphi,D} : C^*(V_1, \ldots, V_n) \to B(\mathcal{H})$, given by $\Psi_{\varphi,D}(X) := D^{1/2} \Psi(X) D^{1/2}$, is a completely positive linear map such that $\Psi_{\varphi,D}(I) = D$ and

$$\Psi_{\varphi,D}(V_\alpha V_\beta^*) = D^{1/2} T_\alpha T_\beta^* D^{1/2} = A_\alpha D A_\beta^*, \quad \alpha, \beta \in \mathbb{F}_n^+.$$

Therefore, the map $\Phi_{\varphi,D}$ has the required properties and $\|\Phi_{\varphi,D}\|_{cb} \leq \|D\|$.

Now, let us assume that $\varphi(I) = I$ and D is only a positive operator such that $\varphi(D) = D$. Hence, if $\epsilon > 0$, then $D + \epsilon I$ is positive invertible and $\varphi(D + \epsilon I) = D + \epsilon I$. Applying the first part of the theorem, we find a completely positive linear map $\Psi_\epsilon : \mathcal{O}_n \to B(\mathcal{H})$ such that

$$\Psi_\epsilon(v_\alpha v_\beta^*) = A_\alpha D A_\beta + \epsilon A_\alpha A_\beta, \quad \alpha, \beta \in \mathbb{F}_n^+, \tag{2.11}$$

and

$$\|\Psi_\epsilon\|_{cb} \leq \|D + \epsilon I\|. \tag{2.12}$$

If $q(v_1, \ldots, v_n) := \sum_{\alpha,\beta \in \mathbb{F}_n^+} a_{\alpha\beta} v_\alpha v_\beta^*$ is a polynomial in $\mathcal{O}_n$, define

$$q^D(A_1, \ldots, A_n) := \sum_{\alpha,\beta \in \mathbb{F}_n^+} a_{\alpha\beta} A_\alpha D A_\beta^*.$$

The definition is correct since, according to (2.12), we have

$$\|q^D(A_1, \ldots, A_n)\| \leq \|D\| \|q(V_1, \ldots, V_n)\|. \tag{2.13}$$

Define $\Phi_{\varphi,D} : \mathcal{O}_n \to B(\mathcal{H})$ by setting $\Phi_{\varphi,D} f := f^D(A_1, \ldots, A_n)$, $f \in \mathcal{O}_n$, where

$$f^D(A_1, \ldots, A_n) := \lim_{k \to \infty} q_k^D(A_1, \ldots, A_n) \tag{2.14}$$

and $q_k(v_1, \ldots, v_n)$ is an arbitrary sequence of polynomials in $\mathcal{O}_n$ convergent to f. Using relation (2.11) and standard approximation arguments, we deduce

$$\Psi_\epsilon(f) = \Phi_{\varphi,D}(f) + \epsilon P_{\varphi,I}(f), \quad f \in \mathcal{O}_n,$$

where $P_{\varphi,I}$ is the Poisson transform associated with φ and I. Taking $\epsilon \to 0$, we infer that

$$\Phi_{\varphi,D}(f) = \lim_{\epsilon \to 0} \Psi_\epsilon(f), \quad f \in \mathcal{O}_n,$$

in the uniform norm. Since for each $\epsilon > 0$, Ψ_ϵ is a completely positive linear map satisfying (2.12), we deduce that $\Phi_{\varphi,D}$ is completely positive and $\|\Phi_{\varphi,D}\|_{cb} \leq \|D\|$. The proof is complete. $\square$

3. Ergodic theory of completely positive maps on $B(\mathcal{H})$

In this section we present canonical decompositions (see Theorem 3.1), ergodic type results (see Theorem 3.2), and lifting theorems (see Theorem 3.5) for w^*-continuous positive linear maps on $B(\mathcal{H})$. These results and the Poisson kernels of Section 2 are used to prove the main result of this section (see Theorem 3.8), which provides a complete description of the $C(\varphi)$-sets, when φ is a w^*-continuous completely positive linear map with $\varphi(I) \leq I$. When we drop the condition $\varphi(I) \leq I$, we also obtain characterizations of the $C(\varphi)$-sets (see Theorem 3.4 and Theorem 3.7).

Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$. An operator $C \in B(\mathcal{H})$ is called pure solution of the inequality $\varphi(X) \leq X$ if

$$\text{SOT} - \lim_{k\to\infty} \varphi^k(C) = 0.$$

Notice that a pure solution is always a positive operator. In what follows we present a *canonical decomposition* for the selfadjoint solutions of the operator inequality $\varphi(X) \leq X$.

Theorem 3.1. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$ and let $A \in B(\mathcal{H})$ be a selfadjoint solution of the inequality $\varphi(X) \leq X$. Then there exist operators $B, C \in B(\mathcal{H})$ with the properties:*

(i) $B = B^*$ *is a solution of the equation* $\varphi(X) = X$*;*
(ii) $C \geq 0$ *is a pure solution of the inequality* $\varphi(X) \leq X$*;*
(iii) $A = B + C$.

Moreover, this decomposition is unique.

Proof. The sequence of selfadjoint operators $\{\varphi^k(A)\}_{k=0}^\infty$ is bounded and decreasing. Thus it converges strongly to to a selfadjoint operator $B := \text{SOT}-\lim_{k\to\infty} \varphi^k(A)$. Since φ is w^*-continuous, we have $\varphi(B) = B$. Setting $C := A - B$, we clearly have $C \geq 0$ and

$$\varphi(C) = \varphi(A) - B \leq A - B = C.$$

Since $\varphi^k(C) \to 0$ strongly, as $k \to \infty$, C is a pure solution of the inequality $\varphi(X) \leq X$.

Now suppose $A = B_1 + C_1$ with $\varphi(B_1) = B_1$ and C_1 is a pure solution of the inequality $\varphi(X) \leq X$. Then

$$B - B_1 = \varphi^k(B - B_1) = \varphi^k(C_1 - C) \to 0, \text{ as } k \to \infty.$$

Therefore, $B = B_1$ and $C = C_1$. The proof is complete. □

Let us remark that a result similar to Theorem 3.1 holds if A is a selfadjoint solution of the inequality $\varphi(X) \geq X$.

Now, we can prove the following ergodic type result.

Theorem 3.2. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$ and let $A \in B(\mathcal{H})$ be a selfadjoint solution of the inequality $\varphi(X) \leq X$. Then*

$$\text{SOT} - \lim_{k\to\infty} \frac{\varphi^0(A) + \varphi^1(A) + \cdots + \varphi^{k-1}(A)}{k} = B,$$

where $A = B + C$ is the canonical decomposition of A with respect to φ, and $\varphi(B) = B$.

Proof. Since C is a pure solution of the inequality $\varphi(X) \leq X$, we have $\text{SOT} - \lim_{k\to\infty} \varphi^k(C) = 0$. Taking into account that $0 \leq \varphi^k(C) \leq C$, $k = 0, 1, \ldots$, a standard argument shows that

$$\text{SOT} - \lim_{k\to\infty} \frac{\varphi^0(C) + \varphi^1(C) + \cdots + \varphi^{k-1}(C)}{k} = 0.$$

On the other hand, since $A = B + C$ and $\varphi(B) = B$, we infer that

$$\frac{\varphi^0(A) + \varphi^1(A) + \cdots + \varphi^{k-1}(A)}{k} = B + \frac{\varphi^0(C) + \varphi^1(C) + \cdots + \varphi^{k-1}(C)}{k}.$$

Hence, the result follows. □

We recall from [30] the following Wold type decomposition for isometries with orthogonal ranges. Let $V_i \in B(\mathcal{K})$, $i = 1, \ldots, n$, be isometries with $V_i^* V_j = 0$ if $i \neq j$. Then there are subspaces $\mathcal{K}_c, \mathcal{K}_s \subseteq \mathcal{K}$ reducing for each $V_1, \ldots, V_n$, such that

(i) $\mathcal{K} = \mathcal{K}_c \oplus \mathcal{K}_s$;
(ii) $\left(\sum_{i=1}^n V_i V_i^*\right) |\mathcal{K}_c = I_{\mathcal{K}_c}$;
(iii) $\{V_i|\mathcal{K}_s\}_{i=1}^n$ is unitarily equivalent to $\{S_i \otimes I_{\mathcal{M}}\}_{i=1}^n$ for some Hilbert space $\mathcal{M}$.

Moreover, the decomposition is unique, up to a unitary equivalence. Since the isometries $\{V_i|\mathcal{K}_c\}_{i=1}^n$ generate a representation of the Cuntz algebra $\mathcal{O}_n$, we call $\mathcal{K}_c$ the Cuntz part in the Wold decomposition $\mathcal{K} = \mathcal{K}_c \oplus \mathcal{K}_s$.

Corollary 3.3. *Let φ_T be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ such that $\varphi_T(I) \leq I$ and $\varphi_T(X) = \sum_{i=1}^n T_i X T_i^*$ ($n \in \mathbb{N}$ or $n = \infty$). Then*

$$\text{SOT} - \lim_{k\to\infty} \frac{\varphi_T^0(I) + \varphi_T^1(I) + \cdots + \varphi_T^{k-1}(I)}{k} = P_{\mathcal{H}} P_{\mathcal{K}_c} |\mathcal{H},$$

where $\mathcal{K}_c$ is the Cuntz part in the Wold decomposition $\mathcal{K} = \mathcal{K}_c \oplus \mathcal{K}_s$ of the minimal isometric dilation $[V_1, \ldots, V_n]$ of $[T_1, \ldots, T_n]$ on the Hilbert space $\mathcal{K} \supseteq \mathcal{H}$.

Proof. Since $[V_1, \ldots, V_n]$ is the minimal isometric dilation of $[T_1, \ldots, T_n]$ on $\mathcal{K} \supseteq \mathcal{H}$, we have $V_i^*|\mathcal{H} = T_i^*$, $i = 1, \ldots, n$. Define the completely positive map $\varphi_V(Y) := \sum_{i=1}^n V_i Y V_i^*$, $Y \in B(\mathcal{K})$. Since $\varphi_T^j(I_{\mathcal{H}}) = P_{\mathcal{H}} \varphi_V^j(I_{\mathcal{K}})|\mathcal{H}$, $j = 1, 2, \ldots$, it remains to prove that

$$\text{SOT} - \lim_{k\to\infty} \frac{\varphi_V^0(I) + \varphi_V^1(I) + \cdots + \varphi_V^{k-1}(I)}{k} = P_{\mathcal{K}_c}. \tag{3.1}$$

According to Theorem 3.2 and Theorem 3.1, the limit in (3.1) is equal to $\text{SOT} - \lim_{k\to\infty} \varphi_V^k(I_{\mathcal{K}})$. The noncommutative Wold decomposition [30], shows that

$$\text{SOT} - \lim_{k\to\infty} \varphi_V^k(I_{\mathcal{K}}) = P_{\mathcal{K}_c}.$$

This completes the proof. □

We recall [30] that an n-tuple of operators $[T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, is a C_0-row contraction if $T_1 T_1^* + \cdots + T_n T_n^* \leq I_{\mathcal{H}}$ and

$$\lim_{k\to\infty} \sum_{\alpha \in \mathbb{F}_n^+, |\alpha| = k} \|T_\alpha^* h\|^2 = 0 \quad \text{for any } h \in \mathcal{H}.$$

In what follows, we obtain a characterization of the solutions of the inequality $\varphi(X) \leq X$ (resp. equation $\varphi(X) = X$), where φ is a w^*-continuous completely positive linear map on $B(\mathcal{H})$.

Theorem 3.4. *Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ given by*

$$\varphi(X) := \sum_{i=1}^{n} A_i X A_i^*, \quad X \in B(\mathcal{H}).$$

A positive operator $C \in B(\mathcal{H})$ is a solution of the inequality $\varphi(X) \leq X$ (resp. equation $\varphi(X) = X$) if and only if there exist operators $B_i \in B(\mathcal{H})$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n B_i B_i^* \leq 1$ (resp. $\sum_{i=1}^n B_i B_i^* = 1$) and

$$A_i C^{1/2} = C^{1/2} B_i, \quad i = 1, \ldots, n. \tag{3.2}$$

Moreover, C is a pure solution of $\varphi(X) \leq X$ if and only if there exists a C_0-row contraction $[B_1, \ldots, B_n]$ satisfying relation (3.2)*.*

Proof. Assume that $C \in B(\mathcal{H})$ is a solution of the inequality $\varphi(X) \leq X$ (resp. equation $\varphi(X) = X$). Define the operator $G_i : \overline{\text{range } C^{1/2}} \to \overline{\text{range } C^{1/2}}$ by setting

$$G_i^* C^{1/2} := C^{1/2} A_i^*, \quad i = 1, \ldots, n.$$

The definition is correct since

$$\sum_{i=1}^{n} \|G_i^* C^{1/2} h\|^2 = \sum_{i=1}^{n} \|C^{1/2} A_i^* h\|^2 = \langle \varphi(C)h, h\rangle \leq \|C^{1/2} h\|^2. \tag{3.3}$$

If $\varphi(C) = C$, then we have equality in (3.3). Let Q_i, $i = 1, \ldots, n$, be bounded operators on $\mathcal{M} := (\overline{\text{range } C^{1/2}})^\perp$ such that $\sum_{i=1}^n Q_i Q_i^* = I$. Define $B_i := G_i \oplus Q_i$, $i = 1, \ldots, n$, with respect to the decomposition $\mathcal{H} = \mathcal{M}^\perp \oplus \mathcal{M}$, and notice that $\sum_{i=1}^n B_i B_i^* \leq I$ if $\varphi(C) \leq C$, and $\sum_{i=1}^n B_i B_i^* = I$ if $\varphi(C) = C$.

Conversely, assume that $B_i \in B(\mathcal{H})$ satisfies $A_i C^{1/2} = C^{1/2} B_i$, for any $i = 1, \ldots, n$. Then we have

$$\varphi(C) = C^{1/2} \left(\sum_{i=1}^{n} B_i B_i^* \right) C^{1/2} \leq C$$

if $\sum\limits_{i=1}^{n} B_i B_i^* \leq I$, and $\varphi(C) = C$ if $\sum\limits_{i=1}^{n} B_i B_i^* = I$.

To prove the second part of the theorem, assume that C is a pure solution of $\varphi(X) \leq X$. Following the first part of the proof, define $B_i := G_i \oplus 0$, $i = 1, \ldots, n$, with respect to the decomposition $\mathcal{H} = \mathcal{M}^\perp \oplus \mathcal{M}$. Since

$$\sum_{|\alpha|=k} \|G_\alpha^* C^{1/2} h\|^2 = \langle \varphi^k(C)h, h\rangle, \quad h \in \mathcal{H},$$

it is clear that $\sum\limits_{|\alpha|=k} B_\alpha B_\alpha^* \to 0$ strongly, as $k \to 0$. Therefore, $[B_1, \ldots, B_n]$ is a C_0-row contraction. For the converse, it is enough to observe that

$$\varphi^k(C) = C^{1/2} \left(\sum_{|\alpha|=k} B_\alpha B_\alpha^* \right) C^{1/2}.$$

This completes the proof. $\square$

Consider now the case when φ is a w^*-continuous completely positive linear map on $B(\mathcal{H})$ with $\varphi(I)\le I$. Let

$$\varphi_T(X):=\sum_{i=1}^n T_iXT_i^*,\quad X\in B(\mathcal{H}),$$

where $n\in\mathbb{N}$ or $n=\infty$, and let

$$\varphi_V(Y):=\sum_{i=1}^n V_iYV_i^*,\quad Y\in B(\mathcal{K}),$$

where $[V_1,\dots,V_n]$ is the minimal isometric dilation of the row contraction $[T_1,\dots,T_n]$ on the Hilbert space $\mathcal{K}\supseteq\mathcal{H}$ (see [30]), i.e., $V_i^*|_{\mathcal{H}}=T_i^*,\ i=1,\dots,n$, and $\mathcal{K}=\bigvee_{\alpha\in\mathbb{F}_n^+}V_\alpha\mathcal{H}$. We call φ_V the minimal dilation of φ_T. Notice that φ_V is a normal $*$-endomorphism of $B(\mathcal{H})$ such that

$$\langle\varphi_T^k(X)h,h'\rangle=\langle\varphi_V^k(X)h,h'\rangle,$$

for any $h,h'\in\mathcal{H},\ \ X\in B(\mathcal{H})\subseteq B(\mathcal{K})$, and $k\in\mathbb{N}$. Here we identify $X\in B(\mathcal{H})$ with $P_{\mathcal{H}}XP_{\mathcal{H}}\in B(\mathcal{K})$, where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

The noncommutative dilation theory together with Theorem 3.4 can be used to obtain the following lifting theorem for the solutions of the operator inequality $\varphi_T(X)\le X$ (resp. equation $\varphi_T(X)=X$).

Theorem 3.5. *Let φ_T be a w^*-continuous completely positive linear map with $\varphi_T(I)\le I$ and let φ_V be its minimal isometric dilation.*

(i) *A positive operator $C\in B(\mathcal{H})$ is a solution of the inequality $\varphi_T(X)\le X$ (resp. equation $\varphi_T(X)=X$), if and only if $C:=P_{\mathcal{H}}D|\mathcal{H}$, where D is a positive solution of the inequality $\varphi_V(Y)\le Y$ (resp. equation $\varphi_V(Y)=Y$), such that $\|C\|=\|D\|$.*

(ii) *An operator $C\in B(\mathcal{H})$ is a pure solution of the inequality $\varphi_T(X)\le X$ if and only if $C:=P_{\mathcal{H}}D|\mathcal{H}$, where D is a pure solution of the inequality $\varphi_V(Y)\le Y$, such that $\|C\|=\|D\|$.*

Proof. Assume that $C\in B(\mathcal{H})$ is a solution of the inequality $\varphi_T(X)\le X$. Taking into account Theorem 3.4, we find $B_i\in B(\mathcal{H})$ satisfying

$$T_iC^{1/2}=C^{1/2}B_i,\ i=1,\dots,n,$$

where $[B_1,\dots,B_n]$ is a row contraction which has the property $\sum_{i=1}^n B_iB_i^*=1$ if $\varphi(C)=I$, and $\sum\limits_{|\alpha|=k}B_\alpha B_\alpha^*\to 0$ strongly, as $k\to 0$, if $\varphi^k(C)\to 0$. Let $[V_1,\dots,V_n]$ be the minimal isometric dilation of $[T_1,\dots,T_n]$ on a Hilbert space $\mathcal{K}_1\supseteq\mathcal{H}$, and let $[W_1,\dots,W_n]$ be the minimal isometric dilation of $[B_1,\dots,B_n]$ on a Hilbert space $\mathcal{K}_2\supseteq\mathcal{H}$. According to the noncommutative commutant lifting theorem [30] (see also [34]), there exists an operator $\tilde{C}:\mathcal{K}_1\to\mathcal{K}_2$ such that $C^{1/2}=\tilde{C}|\mathcal{H},\ \|\tilde{C}\|=\|C^{1/2}\|$, and

$$\tilde{C}V_i^*=W_i\tilde{C},\quad i=1,\dots,n.$$

Notice that

$$\varphi_V(\tilde{C}^*\tilde{C})=\tilde{C}^*\left(\sum_{i=1}^n W_iW_i^*\right)\tilde{C}\le\tilde{C}^*\tilde{C},$$

if $\varphi_T(C)\le C$ and $\varphi_V(\tilde{C}^*\tilde{C})=\tilde{C}^*\tilde{C}$ if $\varphi_T(C)=C$. Setting $D:=\tilde{C}^*\tilde{C}$, we have $\|D\|=\|C\|$, $C=P_{\mathcal{H}}D|\mathcal{H}$, and the first part of the theorem is proved.

For the second part, if $\varphi_T^k(C) \to 0$ strongly, as $k \to \infty$, then, according to Theorem 3.4, the row-contraction $[B_1, \dots, B_n]$ is of class C_0 and its minimal isometric dilation $[W_1, \dots, W_n]$ is a C_0-row isometry. Therefore,

$$\varphi_V^k(D) = \tilde{C}^* \left(\sum_{|\alpha|=k} W_\alpha W_\alpha^* \right) \tilde{C} \to 0 \text{ strongly, as } k \to \infty.$$

Conversely, if D is a solution of the inequality $\varphi_V(Y) \le Y$, then

$$\begin{aligned}\varphi(P_{\mathcal{H}} D|\mathcal{H}) &= \sum_{i=1}^n T_i (P_{\mathcal{H}} D|\mathcal{H}) T_i^* = P_{\mathcal{H}} (\sum_{i=1}^n V_i D V_i^*)|\mathcal{H} \\ &= P_{\mathcal{H}} \varphi_V(D)|\mathcal{H} \le P_{\mathcal{H}} D|\mathcal{H}.\end{aligned}$$

Notice that we have equality if $\varphi_V(D) = D$. On the other hand, since

$$\varphi^k(P_{\mathcal{H}} D|\mathcal{H}) = P_{\mathcal{H}} \varphi_V^k(D)|\mathcal{H} \to 0 \text{ strongly, as } k \to \infty,$$

it is clear that $C := P_{\mathcal{H}} D|\mathcal{H}$ is a pure solution of the inequality $\varphi_T(X) \le X$ when D is a pure solution of the inequality $\varphi_V(Y) \le Y$. The proof is complete. □

Corollary 3.6. ([13]) *If $\varphi_T(I) = I$, then a positive operator $C \in B(\mathcal{H})$ is a solution of the equation $\varphi_T(X) = X$ if and only if there exists $D \in \{V_i, V_i^*\}'$ such that*

$$C = P_{\mathcal{H}} D|\mathcal{H}, \ \|D\| = \|C\|.$$

Moreover, the result remains true if C is a selfadjoint operator.

Proof. Notice that if $\varphi_V(Y) = Y$, then $Y \in \{V_i, V_i^*\}'$. If $\varphi_T(I) = I$, then the converse is also true. In this case we have $\sum_{i=1}^n V_i V_i^* = I$. Applying Theorem 3.5, the result follows. □

The Poisson kernels of Section 2 can be used to better understand the structure of the pure solutions of the inequality $\varphi(X) \le X$, where φ is a w^*-continuous completely positive linear map on $B(\mathcal{H})$.

Theorem 3.7. *Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ given by*

$$\varphi(X) := \sum_{i=1}^n A_i X A_i^*, \quad X \in B(\mathcal{H}).$$

A positive operator $C \in B(\mathcal{H})$ is a pure solution of the inequality $\varphi(X) \le X$ if and only if there is a Hilbert space $\mathcal{D}$ and an operator $K : \mathcal{H} \to F^2(H_n) \otimes \mathcal{D}$ such that

$$C = K^* K \quad \textit{and} \quad K A_i^* = (S_i^* \otimes I_{\mathcal{D}}) K, \ i = 1, \dots, n. \tag{3.4}$$

Proof. Assume C is a positive solution of $\varphi(X) \le X$. Let $K_{\varphi,C} : \mathcal{H} \to F^2(H_n) \otimes \mathcal{D}$ be the Poisson kernel associated with φ and C, i.e.,

$$K_{\varphi,C} h := \sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes \Delta A_\alpha^* h, \quad h \in \mathcal{H},$$

where $\Delta := (C - \varphi(C))^{1/2}$ and $\mathcal{D} := \overline{\text{range } \Delta}$. According to the results of Section 2, the relation (3.4) holds when $K := K_{\varphi,C}$.

Conversely, if relation (3.4) is satisfied, then

$$\varphi(K^*K) = \sum_{i=1}^n A_i K^*KA_i^* = K^*\left(\sum_{i=1}^n S_iS_i^* \otimes I\right) K \leq K^*K.$$

Since $\varphi^k(K^*K) = K^*\left(\sum\limits_{|\alpha|=k}^n S_\alpha S_\alpha^* \otimes I\right) K$ is strongly convergent to zero as $k \to \infty$, the result follows. □

The main result of this section is the following theorem which characterizes the positive solutions of the operator inequality $\varphi_T(X) \leq X$, when $\varphi_T(I) \leq I$.

Theorem 3.8. *Let φ_T be a w^*-continuous completely positive linear map with $\varphi_T(I) \leq I$ and let φ_V be its minimal isometric dilation. A positive operator $A \in B(\mathcal{H})$ is a solution of the operator inequality $\varphi_T(X) \leq X$ if and only if there exist*

(i) *an operator $B \in B(\mathcal{K})$ in the commutant of $\{V_i, V_i^*\}_{i=1}^\infty$,*
(ii) *a Hilbert space $\mathcal{D}$, and an operator $K : \mathcal{K} \to F^2(H_n) \otimes \mathcal{D}$ with*

$$V_iK^* = K^*(S_i \otimes I_{\mathcal{D}}), \quad i = 1, 2, \ldots,$$

such that

$$A = P_{\mathcal{H}}A_1|_{\mathcal{H}} + P_{\mathcal{H}}A_2|_{\mathcal{H}}, \quad \|A\| = \|A_1 + A_2\|, \tag{3.5}$$

where

$$A_1 := [\mathrm{SOT} - \lim_{k\to\infty} \varphi_V^k(B)], \quad A_2 := K^*K,$$

and $P_{\mathcal{H}}$ is the orthogonal projection on $\mathcal{H}$. Moreover, the canonical decomposition of A with respect to φ_T coincides with the decomposition from (3.5).

Proof. According to Theorem 3.5, a positive operator $A \in B(\mathcal{H})$ is a solution of the operator inequality $\varphi_T(X) \leq X$ if and only if there exists a positive operator $D \in B(\mathcal{K})$ such that

$$\varphi_V(D) \leq D, \ A = P_{\mathcal{H}}D|\mathcal{H}, \text{ and } \|A\| = \|D\|.$$

Let $D = B + C$ be the canonical decomposition of D with respect to φ_V, i.e.,

(a) $B = B^*$ is a solution of the equation $\varphi_V(Y) = Y$;
(b) $C \in B(\mathcal{K})$ is a pure solution of the inequality $\varphi_V(Y) \leq Y$.

According to Theorem 3.1, we have $B = \mathrm{SOT}-\lim\limits_{k\to\infty} \varphi_V^k(D)$. Since $\varphi_V(B) = B$, it is clear that B in the commutant of $\{V_i, V_i^*\}_{i=1}^\infty$, and $B = \varphi_V^k(B)$, for any $k \in \mathbb{N}$. On the other hand, applying Theorem 3.7 to the operator C, we infer that there is a Poisson kernel $K : \mathcal{K} \to F^2(H_n) \otimes \mathcal{D}$ associated with φ_V and C, such that

$$KV_i^* = (S_i^* \otimes I)K, \quad i = 1, \ldots, n,$$

and $C = K^*K$. Summing up and setting $A_2 := C$, we obtain relation (3.5).

Conversely, if $B \in B(\mathcal{K})$ in the commutant of $\{V_i, V_i^*\}_{i=1}^\infty$, then $\varphi_V(B) \leq B$. Hence $A_1 := [\mathrm{SOT} - \lim\limits_{k\to\infty} \varphi_V^k(B)]$ exists and $\varphi_V(A_1) = A_1$. On the other hand, if K satisfies (ii), then, according to Theorem 3.7, the operator KK^* is a pure solution of the inequality $\varphi_V(Y) \leq Y$. Therefore, we have $\varphi_V(A_1 + KK^*) \leq A_1 + KK^*$. As in Theorem 3.5, we infer that

$$A = P_{\mathcal{H}}A_1|_{\mathcal{H}} + P_{\mathcal{H}}KK^*|_{\mathcal{H}}$$

is a positive solution of the inequality $\varphi_T(X) \leq X$. The proof is complete. □

Corollary 3.9. *If $\varphi_T(I) \leq I$ and $A \in B(\mathcal{H})$ is a positive operator, then $\varphi_T(A) = A$ if and only if there exists $B \in B(\mathcal{K})$, $B \geq 0$, in the commutant of $\{V_i, V_i^*\}_{i=1}^\infty$ such*

$$A = P_\mathcal{H}[\text{SOT} - \lim_{k\to\infty} \varphi_V^k(B)]|_\mathcal{H} = P_\mathcal{H} B P_{\mathcal{K}_u}|_\mathcal{H},$$

where $\mathcal{K}_u$ is the Cuntz part in the Wold decomposition $\mathcal{K} = \mathcal{K}_u \oplus \mathcal{K}_s$ of the minimal isometric dilation $[V_1, \ldots, V_n]$ of $[T_1 \ldots, T_n]$ on the Hilbert space $\mathcal{K} \supseteq \mathcal{H}$.

Let $\mathcal{K}$ and $\mathcal{K}'$ be Hilbert spaces. We recall from [31] that a bounded linear operator $M \in B(F^2(H_n)\otimes\mathcal{K}, F^2(H_n)\otimes\mathcal{K}')$ is *multi-analytic* if $M(S_i \otimes I_\mathcal{K}) = (S_i \otimes I_{\mathcal{K}'})M$, $i = 1, \ldots, n$. The set of multi-analytic operators coincides with the operator space $R_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, where R_n^∞ is the commutant of the noncommutative analytic Toeplitz algebra F_n^∞. More about multi-analytic operators on Fock spaces can be found in [32], [35], and [38].

Corollary 3.10. ([39]) *If $\varphi_T(I) \leq I$ and φ_T is pure, i.e., $\varphi_T^k(I) \to 0$ strongly, as $k \to \infty$, then any positive solution A of the inequality $\varphi_T(X) \leq X$ is pure and $A = P_\mathcal{H}\Psi\Psi^*|_\mathcal{H}$, where Ψ is a multi-analytic operator.*

Proof. Since $\varphi_T^k(A) \leq \|A\|\,\varphi_T^k(I)$, $k = 1, 2, \ldots$, and φ_T is pure, we infer that any positive solution of the inequality $\varphi_T(X) \leq X$ is pure. On the other hand, since φ_T is pure, $[T_1, \ldots, T_n]$ is a C_0-row contraction. According to [30], its minimal isometric dilation ca be identified with $[S_1 \otimes I_\mathcal{M}, \ldots, S_n \otimes I_\mathcal{M}]$ for some Hilbert space $\mathcal{M}$. Applying Theorem 3.8, we have

$$(S_i \otimes I_\mathcal{M})K^* = K^*(S_i \otimes I_\mathcal{D}), \quad i = 1, \ldots, n,$$

i.e., K^* is a multi-analytic operator, and the result follows. □

Proposition 3.11. *If $\varphi_T(I) = I$ and $A \in B(\mathcal{H})$ is a selfadjoint operator, then $\varphi_T(A) \leq A$ if and only if there exist an operator $B \in B(\mathcal{K})$ in the commutant of $\{V_i, V_i^*\}_{i=1}^\infty$, a Hilbert space $\mathcal{D}$, and an operator $K : \mathcal{K} \to F^2(H_n) \otimes \mathcal{D}$ with $V_iK^* = K^*(S_i \otimes I_\mathcal{D})$, $i = 1, \ldots, n$, such that*

$$A = P_\mathcal{H} B|_\mathcal{H} + P_\mathcal{H} KK^*|_\mathcal{H}, \tag{3.6}$$

where $P_\mathcal{H}$ is the orthogonal projection on $\mathcal{H}$. Moreover, the canonical decomposition of A with respect to φ_T coincides with the decomposition (3.6).

Proof. Let $A = R + Q$ be the canonical decomposition of A with respect to φ_T. Then the operator $R := [\text{SOT} - \lim_{k\to\infty} \varphi_T^k(A)]$ is selfadjoint and $\varphi_T(R) = R$, and Q is a pure solution of the inequality $\varphi_T(X) \leq X$. Applying Corollary 3.6 to the operator R, we find $B \in B(\mathcal{K})$ in the commutant of $\{V_i, V_i^*\}_{i=1}^\infty$ such that $R = P_\mathcal{H} B|\mathcal{H}$. Using Theorem 3.7, we infer that $Q = KK^*$, as required. The proof of the converse is the same as the one from Theorem 3.8. □

The following result provides a new insight and an alternative proof of Corollary 3.9, which is based on the Poisson transforms of Section 2 and not on the noncommutative commutant lifting theorem.

Proposition 3.12. *Let φ_T be a w^*-continuous completely positive linear map with $\varphi_T(I) \leq I$ and let φ_V be its minimal isometric dilation. If $R \in B(\mathcal{H})$, $R \geq 0$, is a solution of the equation $\varphi_T(X) = X$, then there is $A \in B(\mathcal{K})$ in the commutant of $\{V_i, V_i^*\}_{i=1}^\infty$ such that $R = P_\mathcal{H} A|\mathcal{H}$. Conversely, if $A \in B(\mathcal{K})$ is such that $\varphi_V(A) = A$ then $R := P_\mathcal{H} A|\mathcal{H}$ is a solution of the equation $\varphi_T(X) = X$*

Moreover, if $\varphi_T(I) = I$, then the result remains true if $R \in B(\mathcal{H})$ is a selfadjoint solution of the equation $\varphi_T(X) = X$.

Proof. Let $R \in B(\mathcal{H})$ be such that $0 \le R \le I$ and $\varphi_T(R) = R$. According to Theorem 2.1 there is a unique completely positive linear map $P_{\varphi,R} : C^*(S_1,\dots,S_n) \to B(\mathcal{H})$ such that $P_{\varphi,R}(S_\alpha S_\beta^*) = T_\alpha R T_\beta^*$, $\alpha,\beta \in \mathbb{F}_n^+$, and $P_{\varphi,R}(I) = R$. Since $\varphi(I-R) = \varphi(I) - R \le I - R$, we can apply again Theorem 2.1 and find a completely positive linear map $P_{\varphi,I-R} : C^*(S_1,\dots,S_n) \to B(\mathcal{H})$ such that $P_{\varphi,I-R}(S_\alpha S_\beta^*) = T_\alpha(I-R)T_\beta^*$, $\alpha,\beta \in \mathbb{F}_n^+$, and $P_{\varphi,I-R}(I) = I - R$. Hence, $P_{\varphi,I-R} = P_{\varphi,I} - P_{\varphi,R}$ is a completely positive linear map, where $P_{\varphi,I}$ is the Poisson transform associated with the row contraction $[T_1,\dots,T_n]$. Therefore,

$$0 \le P_{\varphi,R} \le P_{\varphi,I}. \tag{3.7}$$

Notice that $P_{\varphi,I}(x) = P_\mathcal{H}\pi(x)|\mathcal{H}$, where π is the representation of $C^*(S_1,\dots,S_n)$ generated by the minimal isometric dilation $[V_1,\dots,V_n]$ on the Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, i.e., $\pi(S_\alpha S_\beta^*) = V_\alpha V_\beta^*$. On the other hand, since $P_{\varphi,R}$ is a completely positive linear map, according to Stinespring theorem [43], there is a representation $\rho : C^*(S_1,\dots,S_n) \to B(\mathcal{G})$ and an operator $W \in B(\mathcal{H},\mathcal{G})$ such that $P_{\varphi,R}(x) = W^*\rho(x)W$, $x \in C^*(S_1,\dots,S_n)$, and $\bigvee_{\alpha,\beta\in\mathbb{F}_n^+} \rho(S_\alpha S_\beta^*)W\mathcal{H} = \mathcal{G}$. Define the operator $C \in B(\mathcal{K},\mathcal{G})$ be setting

$$C\left(\sum_{i=1}^k V_{\alpha_i}V_{\beta_i}^* h_{\alpha_i\beta_i}\right) := \sum_{i=1}^k \rho(S_{\alpha_i}S_{\beta_i}^*)W h_{\alpha_i\beta_i},$$

where $\alpha_i,\beta_i \in \mathbb{F}_n^+$, $h_{\alpha_i\beta_i} \in \mathcal{H}$, and $k \in \mathbb{N}$. Using (3.7), we have

$$\begin{aligned}
\|\sum_{i=1}^k \rho(S_{\alpha_i}S_{\beta_i}^*)W h_{\alpha_i\beta_i}\|^2 &= \sum_{i,j=1}^k \left\langle P_{\varphi,R}(S_{\beta_j}S_{\alpha_j}^* S_{\alpha_i}S_{\beta_i}^*)h_{\alpha_i\beta_i}, h_{\alpha_j\beta_j}\right\rangle \\
&\le \sum_{i,j=1}^k \left\langle P_{\varphi,I}(S_{\beta_j}S_{\alpha_j}^* S_{\alpha_i}S_{\beta_i}^*)h_{\alpha_i\beta_i}, h_{\alpha_j\beta_j}\right\rangle \\
&= \|\sum_{i=1}^k V_{\alpha_i}V_{\beta_i}^* h_{\alpha_i\beta_i}\|^2.
\end{aligned}$$

This shows that C is well-defined and can be extended to a contraction from $\mathcal{K}$ to $\mathcal{G}$ with the properties $Ch = Wh, h \in \mathcal{H}$ and

$$CV_\alpha V_\beta^* h = \rho(S_\alpha S_\beta^*)Wh, \quad \alpha,\beta \in \mathbb{F}_n^+, h \in \mathcal{H}. \tag{3.8}$$

Hence, we deduce that

$$CV_\alpha V_\beta^* = \rho(S_\alpha S_\beta^*)C, \quad \alpha,\beta \in \mathbb{F}_n^+ \tag{3.9}$$

Indeed, using (3.8), we have

$$CV_\alpha V_\beta^* V_\gamma h = \rho(S_\alpha S_\beta^* S_\gamma)Wh = \rho(S_\alpha S_\beta^*)CV_\gamma h.$$

Now, setting $B := C^*C$, we infer that $0 \le B \le I$ and

$$\begin{aligned}
BV_\alpha V_\beta^* &= C^*CV_\alpha V_\beta^* = C^*\rho(S_\alpha S_\beta^*)C \\
&= V_\alpha V_\beta^* C^*C = V_\alpha V_\beta^* B.
\end{aligned}$$

Therefore, B is in the commutant of $\{V_i, V_i^*\}_{i=1}^n$. On the other hand, notice that

$$\begin{aligned}
\left\langle P_{\mathcal{H}} B V_\alpha V_\beta^* h, h' \right\rangle &= \left\langle C V_\alpha V_\beta^* h, C h' \right\rangle \\
&= \left\langle \rho(S_\alpha S_\beta^*) C h, C h' \right\rangle \\
&= \left\langle W^* \rho(S_\alpha S_\beta^*) W h, h' \right\rangle \\
&= \left\langle P_{\varphi,R}(S_\alpha S_\beta^*) h, h' \right\rangle,
\end{aligned} \tag{3.10}$$

for any $h, h' \in \mathcal{H}$. Therefore, $P_{\varphi,R}(S_\alpha S_\beta^*) = P_{\mathcal{H}} B V_\alpha V_\beta^* |\mathcal{H}$, $\alpha, \beta \in \mathbb{F}_n^+$. Hence, we have $R = P_{\varphi,R}(I) = P_{\mathcal{H}} B|\mathcal{H}$. Since B commutes with each V_i and V_i^*, we have

$$\varphi_V(B) = B^{1/2} \varphi_V(I) B^{1/2} \leq B$$

if $\varphi_V(I) \leq I$, and $\varphi_V(B) = B$ if $\varphi_V(I) = I$. We recall that $\varphi_V(I) = I$ if and only if $\varphi(I) = I$.

For the converse, assume $A \in B(\mathcal{K})$ satisfies $\varphi_V(A) = A$, and let $R := P_{\mathcal{H}} A|\mathcal{H}$. Taking into accout that $[V_1, \ldots, V_n]$ is the minimal isometric dilation of $[T_1, \ldots, T_n]$, we have

$$\begin{aligned}
\left\langle \varphi(R) h, h' \right\rangle &= \sum_{i=1}^n \left\langle A V_i^* h, V_i^* h' \right\rangle = \left\langle \varphi_V(A) h, h' \right\rangle \\
&= \left\langle A h, h' \right\rangle = \left\langle R h, h' \right\rangle
\end{aligned}$$

for any $h, h' \in \mathcal{H}$.

When $\varphi(I) = I$, the result of the theorem remains true if R is a selfadjoint operator satisfying $\varphi(R) = R$. It is enough to apply the first part of the theorem to the positive operators R_1, R_2 in the Jordan decomposition $R = R_1 - R_2$. Notice that, since $\varphi(I) = I$, we have $\varphi(R_j) = R_j$, $j = 1, 2$. The proof is complete. □

4. Common invariant subspaces for n-tuples of operators

We show that there is a strong connection between the positive solutions of the operator inequality $\varphi(X) \leq X$, where φ is a w^*-continuous completely positive linear map on $B(\mathcal{H})$ defined by $\varphi(X) := \sum_{i=1}^n A_i X A_i^*$, $X \in B(\mathcal{H})$, and the common invariant subspaces for the n-tuple of operators $\{A_i\}_{i=1}^n$. In this direction, we obtain invariant subspace theorems (eg. Theorem 4.3) and Wold type decompositions for w^*-continuous completely positive linear maps on $B(\mathcal{H})$ (eg. Theorem 4.7). The latter results generalize the classical Wold decomposition for isometries, as well as the one obtained in [30] for isometries with orthogonal ranges. As in the previous sections we consider $n \in \mathbb{N}$ or $n = \infty$.

Lemma 4.1. *Let φ_A be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ given by $\varphi_A(X) := \sum_{i=1}^n A_i X A_i^*$. If $X \geq 0$ and $\varphi_A(X) \leq X$, then the subspace* $\ker X$ *is invariant under each A_i^*, $i = 1, \ldots, n$. In particular, if $\mathcal{M}$ is a subspace of $\mathcal{H}$ and $\varphi_A(P_{\mathcal{M}}) \leq P_{\mathcal{M}}$, then $\mathcal{M}$ is invariant under each A_i, $i = 1, \ldots, n$. If $\varphi_A(I) \leq I$ and $\mathcal{M}$ is reducing under each A_i, $i = 1, \ldots, n$, then $\varphi_A(P_{\mathcal{M}}) \leq P_{\mathcal{M}}$.*

Proof. Since $X \geq 0$ and $\varphi_A(X) \leq X$, for any $h \in \ker X$, we have

$$0 \leq \sum_{i=1}^n \langle A_i X A_i^* h, h \rangle \leq \langle X h, h \rangle = 0.$$

Hence $\|X^{1/2}A_i^*h\| = 0$, $i = 1, \ldots, n$, whence $A_i^*h \in \ker X$. As a particular case, if $\mathcal{M}$ is a subspace of $\mathcal{H}$ and $\varphi_A(P_\mathcal{M}) \leq P_\mathcal{M}$, then $\mathcal{M}$ is invariant under each A_i, $i = 1, \ldots, n$. On the other hand, if $\varphi_A(I) \leq I$ and $\mathcal{M}$ is reducing under each A_i, $i = 1, \ldots, n$, then $P_\mathcal{M} A_i = A_i P_\mathcal{M}$ and therefore

$$\varphi_A(P_\mathcal{M}) = P_\mathcal{M}\varphi_A(I)P_\mathcal{M} \leq P_\mathcal{M}.$$

The proof is complete. □

Corollary 4.2. *Let φ_T be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ given by $\varphi_T(X) := \sum_{i=1}^n T_i X T_i^*$ with $\varphi_T(I) \leq I$. If $X \in B(\mathcal{H})$ is a positive operator such that $\|X\| = 1$ and $\varphi_T(X) \geq X$ then the fixed-point set $\{h \in \mathcal{H} : Xh = h\}$ is invariant under each T_i^*, $i = 1, \ldots, n$.*

Proof. Notice that $I - X \geq 0$ and

$$\varphi_T(I - X) = \varphi_T(I) - \varphi_T(X) \leq I - X.$$

Applying Lemma 4.1, to the positive operator $I - X$, the result follows. □

Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$. The orbit of $X \in B(\mathcal{H})$ under the semigroup generated by φ is the sequence

$$\varphi^0(X), \varphi^1(X), \varphi^2(X), \ldots,$$

where $\varphi^0(X) := X$. We say that the orbit of $X \in B(\mathcal{H})$ under φ has a fixed point if there is $h \in \mathcal{H}$ such that

$$\varphi^0(X)h = \varphi^k(X)h, \quad k = 1, 2, \ldots.$$

Theorem 4.3. *Let φ_A be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ given by $\varphi_A(X) := \sum_{i=1}^n A_i X A_i^*$. Assume that there is a positive operator $X \in B(\mathcal{H})$, $X \neq 0$, such that $\varphi_A(X) \leq X$. If one of the following statements holds, then there is a nontrivial invariant subspace under each A_i, $i = 1, \ldots, n$:*

- (i) *X is not injective;*
- (ii) *X is not pure with respect to φ_A and there is $h \in \mathcal{H}$, $h \neq 0$, such that $\lim_{k\to\infty} \varphi_A^k(X)h = 0$;*
- (iii) *$\varphi_T(X) \neq X$, and the orbit of X under φ_A has a nonzero fixed point.*

Proof. Due to Lemma 4.1, if X is not injective, then $(\ker X)^\perp$ is invariant under each each A_i, $i = 1, \ldots, n$. Now assume that (ii) or (iii) holds. Let $X = B + C$ be the canonical decomposition of X with respect to φ_A. According to Theorem 3.1, we have

$$B = \text{SOT} - \lim_{k\to\infty} \varphi_A^k(X), \quad \varphi_A(B) = B,$$

and C is a pure solution of the inequality $\varphi_A(X) \leq X$. By Lemma 4.1, the subspaces $(\ker B)^\perp$ and $(\ker C)^\perp$ are invariant under each each A_i, $i = 1, \ldots, n$. On the other hand, we have

$$\ker B = \{h \in \mathcal{H} : \ \text{SOT} - \lim_{k\to\infty} \varphi_A^k(X)h = 0\}$$

and

$$\ker C = \{h \in \mathcal{H} : \ \text{SOT} - \lim_{k\to\infty} \varphi_A^k(X)h = Xh\}.$$

Since $\varphi_A(X) \leq X$, it is easy to see that

$$\ker C = \{h \in \mathcal{H} : \ \varphi_A^k(X)h = Xh, \ k \in \mathbb{N}\}.$$

Now, notice that the condition (ii) (resp. (iii)) holds, if and only if the the subspace $\ker B$ (resp. $\ker C$) is nontrivial, and the result follows. $\square$

Another consequence of Lemma 4.1 is the following result which was also obtained in [13].

Corollary 4.4. *Let φ_T be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ such that $\varphi_T(I) = I$. Then $\mathcal{M} \subseteq \mathcal{H}$ is is an invariant subspace under each T_i, $i = 1, 2, \ldots$, if and only if $\varphi_T(P_\mathcal{M}) \leq P_\mathcal{M}$. Moreover, $\mathcal{M}$ is reducing under each T_i, $i = 1, 2, \ldots$, if and only if $\varphi_T(P_\mathcal{M}) = P_\mathcal{M}$.*

Proof. According to Lemma 4.1, if $\varphi_T(P_\mathcal{M}) \leq P_\mathcal{M}$, then the subspace $\mathcal{H} \ominus \mathcal{M}$ is invariant under under each T_i^*, $i = 1, 2, \ldots$. Conversely, assume $\mathcal{M}$ is invariant under under each T_i, $i = 1, 2, \ldots$. Then $QT_iQ = QT_i$, where $Q := I - P_\mathcal{M}$. It is easy to see that

$$\varphi_T(Q)Q = \varphi_T(I)Q = Q = Q\varphi_T(I)Q = Q\varphi_T(Q)Q.$$

Since the operators $\varphi_T(Q)$ and $I - Q$ are positive and commuting, we have

$$\varphi_T(Q) - Q = \varphi_T(Q)(I - Q) \geq 0.$$

Hence, $\varphi_T(Q) \geq Q$. Since $\varphi_T(I) = I$, we infer that $\varphi_T(P_\mathcal{M}) \leq P_\mathcal{M}$. The last part of the corollary can be proved in a similar manner. $\square$

The following two propositions are needed to prove our Wold type decomposition theorem for w^*-continuous completely positive linear maps on $B(\mathcal{H})$.

Proposition 4.5. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$ with $\|\varphi\| \leq 1$. Then*

$$\varphi^\infty(I) := \text{SOT} - \lim_{k\to\infty} \varphi^k(I)$$

exists and has the following properties:

(i) $0 \leq \varphi^\infty(I) \leq I$;
(ii) $\varphi(\varphi^\infty(I)) = \varphi^\infty(I)$;
(iii) *If $\varphi^\infty(I) \neq 0$, then $\|\varphi^\infty(I)\| = 1$;*
(iv) *If $\varphi^\infty(I)h \neq 0$, then $\varphi^k(I)h \neq 0$ for any $k \in \mathbb{N}$.*

Proof. The first two statements are particular cases of Theorem 3.1. We prove (iii). For any $h \in \mathcal{H}$, $k \in \mathbb{N}$, we have

$$\langle \varphi^\infty(I)h, h\rangle = \langle \varphi^k(\varphi^\infty(I))h, h\rangle \leq \|\varphi^\infty(I)\|\langle \varphi^k(I)h, h\rangle.$$

Taking the limit as $k \to \infty$, we obtain

$$\langle \varphi^\infty(I)h, h\rangle \leq \|\varphi^\infty(I)\|^2 \langle h, h\rangle.$$

Hence, $\|\varphi^\infty(I)^{1/2}\| \leq \|\varphi^\infty(I)^{1/2}\|^2$. Since $\|\varphi^\infty(I)^{1/2}\|^2 = \|\varphi^\infty(I)\| \leq 1$, we deduce that $\|\varphi^\infty(I)^{1/2}\| = 0$ or $\|\varphi^\infty(I)^{1/2}\| = 1$, which proves (iii). For the proof of (iv), let $h \in \mathcal{H}$ be such that $\varphi^\infty(I)h \neq 0$ and assume that there is $k_0 \in \mathbb{N}$ with $\varphi^{k_0}(I)h \neq 0$. Since

$$0 \leq \langle \varphi^{m+k_0}(I)h, h\rangle \leq \langle \varphi^{k_0}(I)h, h\rangle = 0,$$

for any $m \in \mathbb{N}$, we infer that $\varphi^{m+k_0}(I)h = 0$, $m \in \mathbb{N}$. Hence $\varphi^\infty(I)h = 0$, which is a contradiction. Therefore, $\varphi^k(I)h \neq 0$ for any $k \in \mathbb{N}$. $\square$

Let us remark that if $\varphi(I) \leq I$, then

$$\ker \varphi^\infty(I) = \{h \in \mathcal{H} : \lim_{k\to\infty} \varphi^k(I)h = 0\} \tag{4.1}$$

and

$$\ker(I - \varphi^\infty(I)) = \{h \in \mathcal{H} : \varphi^k(I)h = h,\ k \in \mathbb{N}\}. \tag{4.2}$$

Proposition 4.6. *Let φ be a positive linear map on $B(\mathcal{H})$ with $\|\varphi\| \leq 1$. Then $\mathcal{H}$ admits a decomposition of the form*

$$\mathcal{H} = \mathcal{M} \oplus \ker(I - \varphi^\infty(I)) \oplus \ker \varphi^\infty(I), \tag{4.3}$$

and $\mathcal{M} = \{0\}$ if and only if $\varphi^\infty(I)$ is an orthogonal projection.

Proof. Since

$$\mathcal{H} = \overline{\text{range } \varphi^\infty(I)} \oplus \ker \varphi^\infty(I) \quad \text{and} \quad \ker(I - \varphi^\infty(I)) \subseteq \text{range } \varphi^\infty(I),$$

we obtain relation (4.3). On the other hand, since $\varphi^\infty(I)$ is a positive operator, one can prove that

$$\ker[\varphi^\infty(I) - \varphi^\infty(I)^2] = \ker \varphi^\infty(I) \oplus \ker(I - \varphi^\infty(I)).$$

On the other hand, since $\ker[\varphi^\infty(I) - \varphi^\infty(I)^2] = \mathcal{H}$ if and only if $\varphi^\infty(I)$ is an orthogonal projection, the result follows. $\square$

Now we can obtain the following Wold type decomposition.

Theorem 4.7. *Let φ_A be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ given by $\varphi_A(X) := \sum_{i=1}^\infty A_i X A_i^*$ such that $\|\varphi_A\| \leq 1$. Then $\mathcal{H}$ admits a decomposition of the form*

$$\mathcal{H} = \mathcal{M} \oplus \ker(I - \varphi_A^\infty(I)) \oplus \ker \varphi_A^\infty(I),$$

and the subspaces $\ker(I - \varphi_A^\infty(I))$ and $\ker \varphi_A^\infty(I)$ are invariant under each A_i^*, $i = 1, \ldots, n$. If, in addition, $\varphi_A^\infty(I)$ is an orthogonal projection, then we have

$$\mathcal{H} = \ker(I - \varphi_A^\infty(I)) \oplus \ker \varphi_A^\infty(I), \tag{4.4}$$

and the subspaces $\ker(I - \varphi_A^\infty(I))$ and $\ker \varphi_A^\infty(I)$ are reducing for each A_i, $i = 1, \ldots, n$.

Proof. According to Proposition 4.5, we have $\varphi_A(\varphi_A^\infty(I)) = \varphi_A^\infty(I)$. Using Lemma 4.1 and Corollary 4.2, we infer that the subspaces $\ker \varphi_A^\infty(I)$ and $\ker(I - \varphi_A^\infty(I))$ are invariant under each A_i^*, $i = 1, \ldots, n$. The rest of the proof follows from Proposition 4.6. $\square$

Notice that, in particular, if A_i are isometries with orthogonal ranges, then the decomposition (4.4) coincides with the noncommutative Wold decomposition from [30].

5. Similarity of positive linear maps

The main objectives of this section are to provide necessary and sufficient conditions for a w^*-continuous positive linear map φ on $B(\mathcal{H})$ to be similar to a positive linear map λ on $B(\mathcal{H})$ satisfying one of the following properties:

(i) $\lambda(I) = I$ (see Theorem 5.1);
(ii) $\|\lambda\| < 1$ (see Theorem 5.9);
(iii) λ is a pure completely positive linear map with $\|\lambda\| \leq 1$ (see Theorem 5.11);
(iv) λ is a completely positive linear map with $\|\lambda\| \leq 1$ (see Theorem 5.13).

We show that these similarities are strongly related to the existence of invertible positive solutions of the operator inequality $\varphi(X) \leq X$ or equation $\varphi(X) = X$.

We say that two linear maps $\varphi, \lambda : B(\mathcal{H}) \to B(\mathcal{H})$ are similar if there is an invertible operator $R \in B(\mathcal{H})$ such that

$$\varphi(RXR^*) = R\lambda(X)R^*, \quad \text{for any } X \in B(\mathcal{H}). \tag{5.1}$$

Notice that relation (5.1) is equivalent to

$$\varphi = \psi_R \circ \lambda \circ \psi_R^{-1}, \tag{5.2}$$

where $\psi_R(X) := RXR^*, \ X \in B(\mathcal{H})$.

Theorem 5.1. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$. Then the following statements are equivalent:*

(i) *φ if similar to a w^*-continuous positive linear map λ on $B(\mathcal{H})$, with $\lambda(I) = I$.*
(ii) *There exist positive constants $0 < a \leq b$ such that*
$$aI \leq \frac{\varphi^0(I) + \varphi^1(I) + \cdots + \varphi^{k-1}(I)}{k} \leq bI, \quad k \in \mathbb{N};$$
(iii) *There exist positive constants $0 < a \leq b$ and an invertible positive operator $P \in B(\mathcal{H})$ such that*
$$aI \leq \frac{\varphi^0(P) + \varphi^1(P) + \cdots + \varphi^{k-1}(P)}{k} \leq bI, \quad k \in \mathbb{N};$$
(iv) *There exist positive constants $0 < c \leq d$ such that*
$$cI \leq \varphi^k(I) \leq dI, \quad k \in \mathbb{N};$$
(v) *There exist positive constants $0 < c \leq d$ and an invertible positive operator $R \in B(\mathcal{H})$ such that such that*
$$cI \leq \varphi^k(R) \leq dI, \quad k \in \mathbb{N};$$
(vi) *There exists an invertible positive operator $Q \in B(\mathcal{H})$ such that such that $\varphi(Q) = Q$.*

Moreover, the operator Q can be chosen such that $aI \leq Q \leq bI$.

Proof. First we prove that (i) $\Leftrightarrow$ (vi). Assume (i) holds, i.e., $\varphi(RXR^*) = R\lambda(X)R^*$, where $\lambda(I) = I$ and $R \in B(\mathcal{H})$ is an invertible operator such that $aI \leq RR^* \leq bI$, for some constants $0 < a \leq b$. Setting $X := I$ and $Q := RR^*$, we obtain $\varphi(Q) \leq Q$. Conversely, assume (vi) holds and define

$$\lambda(X) := Q^{-1/2}\varphi(Q^{1/2}XQ^{1/2})Q^{-1/2}, \quad X \in B(\mathcal{H}).$$

It is clear that λ is a w^*-continuous positive linear map with $\lambda(I) = I$. Moreover, we have $\varphi = \psi_R \circ \lambda \circ \psi_R^{-1}$, where $R := Q^{1/2}$.

The implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. We prove that (i) $\Rightarrow$ (iv). Assume $\varphi = \psi_R \circ \lambda \circ \psi_R^{-1}$, with $R \in B(\mathcal{H})$ invertible, and $\lambda(I) = I$. Since all the maps are positive, we have

$$\begin{aligned} \varphi^k(I) &\leq \|\psi_R^{-1}(I)\| \, \psi_R(\lambda^k(I)) \\ &\leq \|\psi_R^{-1}(I)\| \|\psi_R(I)\| \, I = \|R^{-1}\|^2 \|R\|^2 \, I. \end{aligned} \tag{5.3}$$

On the other hand, we have

$$I = (\psi_R^{-1} \circ \varphi^k \circ \psi_R)(I) \leq \|\psi_R(I)\| \, R^{-1} \varphi^k(I) R^{*-1}, \quad k \in \mathbb{N}.$$

Hence, we obtain $RR^* \leq \|\psi_R(I)\| \, \varphi^k(I)$, which implies

$$\varphi^k(I) \geq \frac{1}{\|\psi_R(I)\|} RR^* \geq \frac{1}{\|R\|^2 \|R^{-1}\|^2} \, I. \tag{5.4}$$

Putting together relations (5.3) and (5.4), we deduce

$$\frac{1}{\|R\|^2 \|R^{-1}\|^2} \, I \leq \varphi^k(I) \leq \|R^{-1}\|^2 \|R\|^2 \, I, \quad k \in \mathbb{N},$$

which proves (iv).

It remains to show that (iii) $\Rightarrow$ (vi). Let P be an invertible positive operator such that (iii) holds. Since φ is positive, we have

$$\frac{1}{j+1} \varphi^j(P) \leq \frac{1}{j+1} \sum_{q=0}^{j} \varphi^q(P) \leq bI, \quad j \in \mathbb{N}. \tag{5.5}$$

On the other hand, it is clear that, for any $j \leq k$,

$$\varphi^k(P) = (\varphi^{k-j} \circ \varphi^j)(P) \leq \|\varphi^j(P)\| \varphi^{k-j}(I).$$

Hence, and using (5.5), we infer that

$$\begin{aligned} \varphi^k(P) \sum_{j=0}^{k} \frac{1}{j+1} &\leq \sum_{j=0}^{k} \frac{1}{j+1} \|\varphi^j\| \, \varphi^{k-j}(I) \\ &\leq b \sum_{j=0}^{k} \varphi^j(I). \end{aligned} \tag{5.6}$$

Since P is an invertible positive operator and φ^j is positive, we have $I \leq \|P^{-1}\| P$ and

$$\varphi^j(I) \leq \|P^{-1}\| \varphi^j(P).$$

Hence, and using again the inequalities in (iii), we get

$$\begin{aligned} \sum_{j=0}^{k} \varphi^j(I) &\leq (k+1) \|P^{-1}\| \left(\frac{1}{k+1} \sum_{j=0}^{k} \varphi^j(P) \right) \\ &\leq b(k+1) \|P^{-1}\| \, I. \end{aligned}$$

These inequalities together with (5.6) imply

$$\left(\frac{1}{k} \varphi^k(P) \right) \sum_{j=0}^{k} \frac{1}{j+1} \leq \frac{b^2(k+1)}{k} \|P^{-1}\| \, I,$$

for any $k \in \mathbb{N}$. This implies that

$$\sup_k \left\| \frac{1}{k}\varphi^k(P) \sum_{j=0}^{k} \frac{1}{j+1} \right\| < \infty.$$

Hence, we must have

$$\left\| \frac{1}{k}\varphi^k(P) \right\| \to 0, \quad \text{as } k \to \infty. \tag{5.7}$$

For each $k \geq 1$, we define the operator

$$Q_k := \frac{1}{k}\sum_{j=0}^{k} \varphi^j(P).$$

Since $\{Q_k\}_{k=1}^\infty$ is bounded sequence of operators and the closed unit ball of $B(\mathcal{H})$ is weakly compact, there is a subsequence of $\{Q_k\}_{k=1}^\infty$ weakly convergent to an operator $Q \in B(\mathcal{H})$. Due to (iii), the operator Q_k is positive and satisfies $aI \leq Q_k \leq bI$. Therefore, Q is an invertible positive operator satisfying the same inequalities. Since

$$Q_k - \varphi(Q_k) = \frac{1}{k}P - \frac{1}{k}\varphi^k(P)$$

and taking ito account (5.7), we get $\|Q_k - \varphi(Q_k)\| \to 0$, as $k \to \infty$. Now, using the fact that the w^* and the weak topologies coincide on bounded sets of $B(\mathcal{H})$, and that ψ is w^*-continuous, we infer that $\varphi(Q) = Q$. Therefore, (vi) holds and the proof is complete. $\square$

Let us consider an application of this theorem and the results from Section 2. Let $\{A_i\}_{i=1}^\infty \subset B(\mathcal{H})$ ($n \in \mathbb{N}$ or $n = \infty$) be a sequence of operators such that

$$\varphi_A(X) := \sum_{i=1}^{n} A_i X A_i^*, \quad X \in B(\mathcal{H}),$$

is a w^*-continuous completely positive linear map on $B(\mathcal{H})$. According to Theorem 5.1, if there exist some constants $0 < a \leq b$ such that

$$aI \leq \frac{1}{k}\sum_{j=0}^{k-1}\sum_{|\alpha|=j} A_\alpha A_\alpha^* \leq bI, \quad k \in \mathbb{N}, \tag{5.8}$$

then there is an invertible positive operator $Q \in B(\mathcal{H})$ such that $\varphi_A(Q) = Q$ and $aI \leq Q \leq bI$. Therefore, we can apply Theorem 2.1 and Theorem 2.3 to the map φ_A. In particular one can deduce that

$$\|p(A_1, \ldots, A_n)Q^{1/2}\| \leq \|Q^{1/2}\|\|p(S_1, \ldots, S_n)\|,$$

for any polynomial $p(S_1, \ldots, S_n)$ in the noncommutative disc algebra $\mathcal{A}_n$. Hence, we obtain the inequality

$$\|p(A_1, \ldots, A_n)\| \leq \|Q^{1/2}\|\|Q^{-1/2}\|\|p(S_1, \ldots, S_n)\|, \tag{5.9}$$

which can be extended to matrices over $\mathcal{A}_n$. Therefore, if (5.8) holds, then the homomorphism $\Phi : \mathcal{A}_n \to B(\mathcal{H})$, defined by $\Phi(p) := p(A_1, \ldots, A_n)$, is completely bounded and $\|\Phi\|_{cb} \leq \sqrt{\frac{b}{a}}$. We remark that, the inequality (5.9) remains true if we replace the left creation operators by a set of generators of the Cuntz algebra $\mathcal{O}_n$.

Corollary 5.2. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$ such that*

$$\varphi^\infty(I) := \text{SOT} - \lim_{k\to\infty} \varphi^k(I) \tag{5.10}$$

exists. Then φ is similar to a positive linear map ψ such that $\psi(I) = I$ if and only if $\varphi^\infty(I)$ is invertible.

Proof. Since φ is w^*-continuous and the limit (5.10) exists, we have $\varphi(\varphi^\infty(I)) = \varphi^\infty(I)$. If $\varphi^\infty(I)$ is invertible, then the result follows from Theorem 5.1. Conversely, assume that φ is similar to a positive linear map ψ such that $\psi(I) = I$. Using again Theorem 5.1, we have $cI \le \varphi^k(I) \le dI$ for any $k \in \mathbb{N}$. Hence, $\varphi^\infty(I)$ is invertible. □

Let us remark that the limit in (5.10) exists in the particular case when $\varphi(I) \le I$.

In what follows, we find sufficient conditions for the existence of an injective operator in $C_=(\varphi)$. We say that a positive linear map φ on $B(\mathcal{H})$ is power bounded if there is a constant $M > 0$ such that

$$\|\varphi^k\| \le M, \quad k \in \mathbb{N}.$$

Lemma 5.3. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$ such that φ is power bounded and $\langle \varphi^k(I)h, h\rangle$ does not converge to zero for any $h \in \mathcal{H}$, $h \ne 0$. Then there is an injective positive solution of the equation $\varphi(X) = X$.*

Proof. First let us prove that if $Y \in B(\mathcal{H})$, $Y \ge 0$, then the set

$$\{X \ge 0 :\ \varphi(X) = X\} \cap \overline{\text{conv}}^w\{\varphi^k(Y) :\ k = 0, 1, \dots\} \tag{5.11}$$

is nonempty, where $\overline{\text{conv}}^w$ stands for the weakly closed convex hull. Since $\|\varphi^k\| \le M$, $k \in \mathbb{N}$, the sequence of Cesaro means

$$\sigma_k(Y) := \frac{\varphi^0(Y) + \varphi^1(Y) + \cdots + \varphi^{k-1}(Y)}{k}, \quad k = 1, 2, \dots,$$

is bounded. Therefore, there is a subsequence $\{\sigma_{n_k}(Y)\}_{k=1}^\infty$ weakly convergent to an operator $Z \in \overline{\text{conv}}^w\{\varphi^k(Y) :\ k = 0, 1, \dots\}$. Since φ is w^*-continuous positive linear map on $B(\mathcal{H})$ and

$$\|\varphi(\sigma_k(Y)) - \sigma_k(Y)\| \le \frac{\|Y\|}{k} + \frac{\|\varphi^k(Y)\|}{k} \le \frac{(M+1)\|Y\|}{k},$$

for any $k = 1, 2, \dots$, we infer that $\varphi(Z) = Z$. Hence the set (5.11) is nonempty. Now, set $Y := I_{\mathcal{H}}$ and let Z be in the set (5.11). Let $h \in \mathcal{H}$, $h \ne 0$, and assume that $\langle \varphi^k(I)h, h\rangle$ does not converge to zero, as $k \to \infty$. This implies that there is a constant $C > 0$ such that

$$\langle \varphi^k(I)h, h\rangle \ge C, \quad \text{for any } k = 0, 1, \dots. \tag{5.12}$$

Indeed, if this were not true, then there would exist a subsequence $\{n_k\}$ such that $\langle \varphi^{n_k}(I)h, h\rangle \to 0$, as $k \to \infty$. Notice that

$$\begin{aligned}
\langle \varphi^m(I)h, h\rangle &= \langle \varphi^{n_k}(\varphi^{m-n_k}(I))h, h\rangle \\
&\le \|\varphi^{m-n_k}(I)\| \langle \varphi^{n_k}(I)h, h\rangle \\
&\le M\langle \varphi^{n_k}(I)h, h\rangle.
\end{aligned}$$

Therefore, $\langle \varphi^m(I)h, h\rangle \to 0$ as $m \to \infty$, which is a contradiction. Hence, relation (5.12) holds and we have

$$\langle \sum_{k\ge 0} \gamma_k \varphi^k(I)h, h\rangle \ge C$$

for any finitely supported sequence $\{\gamma_n\}$ of positive numbers with $\sum_{k\geq 0}\gamma_k = 1$. Consequently, $\langle Zh, h\rangle \geq C$, which shows that Z is an injective operator. Moreover, we have

$$\ker Z = \{h:\ \langle \varphi^k(I)h, h\rangle \to 0, \text{ as } k\to\infty\}.$$

The proof is complete. □

Corollary 5.4. *If $\mathcal{H}$ is finite dimensional and φ is a w^*-continuous positive linear map on $B(\mathcal{H})$ such that φ is power bounded and $\langle \varphi^k(I)h, h\rangle$ does not converge to zero for any $h\in\mathcal{H}$, $h\neq 0$, then φ is similar to a positive linear map ψ with $\psi(I) = I$.*

Proof. According to Lemma 5.3, there is an injective positive operator $Z\in B(\mathcal{H})$ such that $\varphi(Z) = Z$. Since $\mathcal{H}$ is finite dimensional, Z is invertible. Using Theorem 5.1, the result follows. □

Now, we present a few results concerning the similarity of positive linear maps with contractive (resp. strictly contractive) ones.

Lemma 5.5. *Let φ be a positive linear map on $B(\mathcal{H})$. Then φ if similar to a positive linear map ψ with $\|\psi\| < 1$ if and only if there is an invertible positive operator $R\in B(\mathcal{H})$ such that $R-\varphi(R)$ is positive and invertible.*

Proof. If Q is an invertible operator such that $\|\psi_Q^{-1}\circ\varphi\circ\psi_Q\| < 1$, then we have

$$\|Q^{-1}\varphi(QQ^*)Q^{*-1}\| < c,$$

for some positive constant $c<1$. Hence, we get $\varphi(QQ^*)\leq c\,QQ^*$. Setting $R := QQ^*$, we have

$$R-\varphi(R) = (1-c)R,$$

which is an invertible positive operator. Conversely, assume that $R\in B(\mathcal{H})$ is an invertible positive operator and

$$R-\varphi(R)\geq bI \tag{5.13}$$

for some constant $b>0$. Let a be such that $0<a<1$ and $a<\frac{b}{\|R\|}$. Since $R\leq \|R\|\,I\leq \frac{b}{a}\,I$, we infer that

$$R-bI\leq R-aR. \tag{5.14}$$

Using relations (5.13) and (5.14), we obtain

$$(1-a)R-\varphi(R)\geq R-bI-\varphi(R)\geq 0.$$

Hence, $\varphi(R)\leq (1-a)R$, which implies $\|\psi_{R^{1/2}}^{-1}\circ\varphi\circ\psi_{R^{1/2}}\| < 1$. This completes the proof. □

Proposition 5.6. *Let φ be a positive linear map on $B(\mathcal{H})$. If there is $m\in\mathbb{N}$ such φ^m is similar to a positive linear map ψ with $\|\psi\|\leq 1$ (resp. $\|\psi\|<1$), then φ is similar to a positive linear map ψ' with $\|\psi'\|\leq 1$ (resp. $\|\psi'\|<1$).*

Proof. If φ^m is similar to a positive linear map ψ with $\|\psi\|\leq 1$, then there exists an invertible positive operator Q such that $\varphi^m(Q)\leq Q$. Denote

$$P := Q+\varphi(Q)+\cdots+\varphi^{m-1}(Q)$$

and notice that P is an invertible positive operator. Moreover, we have

$$\begin{aligned}\varphi(P) &= \varphi(Q)+\varphi^2(Q)+\cdots+\varphi^m(Q)\\ &\leq Q+\varphi(Q)+\cdots+\varphi^{m-1}(Q) = P.\end{aligned}$$

Setting $\lambda(X) := P^{-1/2}\varphi(P^{1/2}XP^{1/2})P^{-1/2}$, the result follows. Notice that if $\|\psi\| < 1$, then, according to Lemma 5.5, the operator $Q - \varphi^m(Q)$ is invertible and positive. Since $P - \varphi(P) = Q - \varphi^m(Q)$, we can apply again Lemma 5.5 to complete the proof. □

Proposition 5.7. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$. If there exist positive constants $0 < a \le b$ and a positive operator $P \in B(\mathcal{H})$ such that*

$$aI \le \sum_{k=0}^{\infty} \varphi^k(P) \le bI, \tag{5.15}$$

then φ if similar to a w^*-continuous positive linear map ψ on $B(\mathcal{H})$, with $\|\psi\| \le 1$. Moreover, if, in addition, P is invertible, then φ if similar to a w^*-continuous positive linear map ψ with $\|\psi\| < 1$.

Proof. Setting $Q_m := \sum_{k=0}^{m-1} \varphi^k(P)$, we have

$$Q_{m+1} = Q_m + \varphi^m(P). \tag{5.16}$$

Since $\{Q_m\}_{m=1}^{\infty}$ is a bounded monotone sequence of positive operators, it converges strongly to an operator Q. Due to (5.15), Q is an invertible positive operator. According to (5.16), $\varphi^m(P) \to 0$ strongly, as $m \to \infty$. Since

$$Q_m - \varphi(Q_m) = P - \varphi^m(P),$$

we get $Q - \varphi(Q) = P \ge 0$. If P is invertible, then we can apply Lemma 5.5 to complete the proof. □

Let φ be a positive linear map on $B(\mathcal{H})$. The spectral radius of φ is defined by setting

$$\mathrm{r}(\varphi) := \lim_{k\to\infty} \|\varphi^k\|^{\frac{1}{k}}.$$

Lemma 5.8. *Let φ be a positive linear map on $B(\mathcal{H})$. Then the following statements are equivalent:*

(i) $\mathrm{r}(\varphi) < 1$*;*
(ii) $\lim\limits_{k\to\infty} \|\varphi^k\| = 0$*;*
(iii) $\sum_{k=1}^{\infty} \|\varphi^k\|^p$ *is convergent for any $p > 0$.*

Proof. If (i) holds, then for any $a \in (r(\varphi), 1)$ there is $m \in \mathbb{N}$ such that $\|\varphi^k\| \le a^k$ for any $k \ge m$. this clearly implies conditions (ii) and (iii). Since

$$r(\varphi^n) = \lim_{k\to\infty} \left(\|\varphi^{nk}\|^{\frac{1}{nk}}\right)^n = r(\varphi)^n$$

and $r(\varphi^n) \le \|\varphi^n\|$ for any $n \in \mathbb{N}$, it is clear that (iii) implies (i). □

Now we can characterize those w^*-continuous positive linear maps on $B(\mathcal{H})$ which are similar to strictly contractive ones.

Theorem 5.9. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$. Then the following statements are equivalent:*

(i) *φ if similar to a w^*-continuous positive linear map ψ on $B(\mathcal{H})$, with $\|\psi\| < 1$.*

(ii) *For any invertible positive operator $R \in B(\mathcal{H})$ the equation*

$$X - \varphi(X) = R \tag{5.17}$$

has an invertible positive solution in $B(\mathcal{H})$.

(iii) *There esists an invertible positive operator $Q \in B(\mathcal{H})$ such that $\varphi(Q) \le Q$ and $Q - \varphi(Q)$ is invertible.*

(v) $\lim\limits_{k\to\infty} \|\varphi^k(I)\| = 0$.

(vi) $r(\varphi) < 1$.

Moreover, in this case the positive solution of the equation (5.17) *is unique and given by*

$$X = \sum_{k=0}^{\infty} \varphi^k(R),$$

where the convergence is in the uniform topology.

Proof. The equivalence (i) $\Leftrightarrow$ (iii) (resp. (v) $\Leftrightarrow$ (vi)) was proved in Lemma 5.5 (resp. Lemma 5.8). In what follows we prove that (ii) $\Rightarrow$ (i) $\Rightarrow$ (vi) $\Rightarrow$ (ii). Assume (ii) holds. Let $Q \in B(\mathcal{H})$ be an invertible positive operator such that $Q - \varphi(Q) = R$. Using Lemma 5.5, we infer (i). Now, we assume (i). Then there is an invertible operator Q such that $\|\psi_Q^{-1} \circ \varphi \circ \psi_Q\| < 1$. On the other hand,

$$r(\varphi) = r(\psi_Q^{-1} \circ \varphi \circ \psi_Q) \le \|\psi_Q^{-1} \circ \varphi \circ \psi_Q\| < 1.$$

According to Lemma 5.8, the latter condition is equivalent to the fact that the series $\sum_{k=1}^{\infty} \|\varphi^k\|$ is convergent. Then, for any invertible positive operator $R \in B(\mathcal{H})$, we have

$$\frac{1}{\|R^{-1}\|} I \le R \le \sum_{k=0}^{\infty} \varphi^k(R) \le \left(\|R\| \sum_{k=0}^{\infty} \|\varphi^k\| \right) I.$$

According to Proposition 5.7 (see the proof), there is an invertible operator Q such that $Q - \varphi(Q) = R$, so that (ii) holds.

To prove the last part of the theorem, let $X \ge 0$ be an invertible operator such $X - \varphi(X) = R$, where $R \ge 0$ is a fixed invertible operator. Let $X_k := \sum_{j=0}^{k-1} \varphi^j(R)$, $k \in \mathbb{N}$. Since

$$\varphi^j(R) = \varphi^j(X) - \varphi^{j+1}(X), \quad j = 0, 1, \dots,$$

we have $X_k = X - \varphi^k(X)$, for any $k \in \mathbb{N}$. Since $\|\varphi^k\| \to 0$, as $k \to \infty$, we have

$$0 \le \|X_k - X\| = \|\varphi^k(X)\| \le \|X\| \|\varphi^k\| \to 0,$$

as $k \to \infty$. Therefore, X_k converges to X uniformly, as $k \to \infty$, and X is the unique solution of the inequality $X - \varphi(X) = R$. The proof is complete. $\square$

Corollary 5.10. *Let φ be a w^*-continuous positive linear map on $B(\mathcal{H})$. Then*

$$\mathrm{r}(\varphi) := \inf_Q \|\psi_Q \circ \varphi \circ \psi_Q^{-1}\|, \tag{5.18}$$

where the infimum is taken over all invertible operators $Q \in B(\mathcal{H})$, and $\psi_Q(X) := QXQ^*$.

Proof. Let $\epsilon > 0$ and denote $\varphi_\epsilon := \frac{1}{r(\varphi)+\epsilon} \cdot \varphi$. Since

$$r(\varphi_\epsilon) = \frac{r(\varphi)}{r(\varphi)+\epsilon} < 1,$$

we can apply Theorem 5.9 to deduce that φ_ϵ is similar to a strictly contractive positive map on $B(\mathcal{H})$. Therefore, there is an invertible operator $R_\epsilon \in B(\mathcal{H})$ such that $\|\psi_{R_\epsilon} \circ \varphi_\epsilon \circ \psi_{R_\epsilon}^{-1}\| < 1$. Hence $\frac{1}{r(\varphi)+\epsilon} \|\psi_{R_\epsilon} \circ \varphi \circ \psi_{R_\epsilon}^{-1}\| < 1$ and we can deduce that

$$\inf_Q \|\psi_Q \circ \varphi \circ \psi_Q^{-1}\| \leq \|\psi_{R_\epsilon} \circ \varphi \circ \psi_{R_\epsilon}^{-1}\| \leq r(\varphi) + \epsilon,$$

for any $\epsilon > 0$. On the other hand, we have

$$r(\varphi) = r(\psi_Q \circ \varphi \circ \psi_Q^{-1}) \leq \|\psi_Q \circ \varphi \circ \psi_Q^{-1}\|$$

for all invertible operators $Q \in B(\mathcal{H})$. Now the equation (5.18) follows and the proof is complete. □

The next result provides necessary and sufficient conditions for a w^*-continuous completely positive linear map to be similar to one which is pure and contractive. We recall that a positive linear map φ on $B(\mathcal{H})$ is pure if $\varphi^k(I) \to 0$ strongly, as $k \to \infty$.

Theorem 5.11. *Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$. The following statements are equivalent:*

(i) *φ is similar to a pure completely positive linear map ψ with $\|\psi\| \leq 1$;*
(ii) *There exist two constants $0 < a \leq b$ and a positive operator $R \in B(\mathcal{H})$ such that*

$$aI \leq \sum_{k=0}^{\infty} \varphi^k(R) \leq bI; \tag{5.19}$$

(iii) *There is an invertible pure solution of the inequality $\varphi(X) \leq X$.*

Proof. Let us prove that (ii) ⇒ (i). Assume that (ii) holds. Since φ is a w^*-continuous completely positive linear map on $B(\mathcal{H})$, there exists a sequence $\{A_i\}_{i=1}^n$ ($n \in \mathbb{N}$ or $n = \infty$) such that $\varphi(X) = \sum_{i=1}^n A_i X A_i^*$. Let $W : \mathcal{H} \to F^2(H_n) \otimes \mathcal{H}$ be defined by

$$Wh := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} e_\alpha \otimes R^{1/2} T_\alpha^* h, \quad , h \in \mathcal{H}.$$

Since $\|Wh\|^2 = \sum_{k=0}^{\infty} \langle \varphi^k(R)h, h\rangle$ and relation (5.19) holds, the range of W is a closed subspace of $F^2(H_n) \otimes \mathcal{H}$. Notice that

$$WA_i = (S_i^* \otimes I_{\mathcal{H}})W, \quad i = 1, \ldots, n,$$

and therefore the range of W is invariant under each $S_i^* \otimes I_{\mathcal{H}}$, $i = 1, \ldots, n$. Since the operator $W : \mathcal{H} \to \operatorname{range} W$ is invertible, we have

$$WA_i^* W^{-1} = (S_i^* \otimes I_{\mathcal{H}})|\operatorname{range} W, \quad i = 1, \ldots, n. \tag{5.20}$$

Hence, φ is similar to φ_T, where $\varphi_T(Y) := \sum_{i=1}^n T_i Y T_i^*$ and

$$T_i := P_{\operatorname{range} W}(S_i \otimes I_{\mathcal{H}})|\operatorname{range} W, \quad i = 1, \ldots, n.$$

We clearly have $\|\varphi_T\| \leq 1$ and $\varphi_T^k(I) \to 0$ strongly, as $k \to \infty$. This proves (i).

Assume now that φ is similar to λ such that $\|\lambda\| \leq 1$ and λ is pure. Hence, there is an invertible operator $Q \in B(\mathcal{H})$ such that $\varphi = \psi_Q \circ \lambda \circ \psi_Q^{-1}$. Therefore, $D := QQ^*$ satisfies $\varphi(D) \leq D$. Since λ is pure and $\psi_Q^{-1} \circ \varphi^k \circ \psi_Q = \lambda^k$, we deduce that

$$\varphi^k(D) = Q\lambda^k(I)Q^* \to 0 \quad \text{strongly, as } k \to \infty.$$

Now, let $R := D - \varphi(D)$ and notice that $\sum_{k=0}^\infty \varphi^k(R) = D$, which is invertible and positive. This implies relation (5.19).

We already proved that (i) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) is easy. Indeed, asuume that $D \geq 0$ is an invertible pure solution of the inequality $\varphi(X) \leq X$. Then $\varphi(D) \leq D$ and, if we define

$$\lambda(X) := D^{-1/2}\varphi(D^{1/2}XD^{1/2})D^{-1/2},$$

we have $\lambda(I) \leq I$ and $\lambda^k(I) = D^{-1/2}\varphi(D)D^{-1/2} \to 0$ strongly, as $k \to \infty$. The proof is complete. $\square$

Corollary 5.12. ([29]) *Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$. If there is a constant $b > 0$ such that*

$$\sum_{k=0}^\infty \varphi^k(I) \leq bI,$$

then φ is similar to a pure completely positive linear map λ with $\|\lambda\| \leq 1$.

Notice that if φ is a positive linear map on $B(\mathcal{H})$ with $\|\varphi\| \leq 1$, then

$$I = \varphi^0(D) + \varphi^1(D) + \cdots + \varphi^{k-1}(D) + \varphi^k(I), \quad k \in \mathbb{N},$$

where $D := I - \varphi(I)$. In what follows, we show that a perturbation of this equality provides a characterization of those w^*-continuous completely positive linear maps on $B(\mathcal{H})$ which are similar to contractive ones.

Theorem 5.13. *Let φ_A be a w^*-continuous completely positive linear map on $B(\mathcal{H})$, given by $\varphi_A(X) := \sum_{i=1}^n A_i X A_i^*$ ($n \in \mathbb{N}$ or $n = \infty$). Then the following statements are equivalent:*

- (i) *φ_A if similar to a w^*-continuous completely positive linear map ψ on $B(\mathcal{H})$, with $\|\psi\| \leq 1$;*
- (ii) *There is an invertible positive operator $R \in B(\mathcal{H})$ such that $\varphi_A(R) \leq R$;*
- (iii) *There exist positive constants $0 < a \leq b$ and a positive operator $D \in B(\mathcal{H})$ such that*

$$aI \leq \varphi_A^0(D) + \varphi_A^1(D) + \cdots + \varphi_A^{k-1}(D) + \varphi_A^k(I) \ \leq bI, \quad k \in \mathbb{N};$$

- (iv) *The map $\Phi : \mathcal{A}_n \to B(\mathcal{H})$ defined by $\Phi(p) := p(A_1, \ldots, A_n)$ is completely bounded, where $\mathcal{A}_n$ is the noncommutative disc algebra.*

Moreover, if (iii) *holds, then $\|\Phi\|_{cb} \leq \sqrt{\frac{b}{a}}$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is clear. Let us prove that (ii) $\Rightarrow$ (iii). Assume that R is an invertible operator such that $0 \leq R \leq I$ and $\varphi_A(R) \leq A$. Let $D := R - \varphi_A(R)$ and notice that

$$\begin{aligned}\varphi_A^0(D) + \varphi_A^1(D) + \cdots + \varphi_A^{k-1}(D) + \varphi_A^k(I) &= R - \varphi_A^k(R) + \varphi_A^k(I)\\ &\geq R \geq \frac{1}{\|R^{-1}\|} I.\end{aligned}$$

On the other hand, since φ_A is similar to a positive linear map of norm less than one, there is a constant $M>0$ such that $\varphi_A^k(I)\leq M$, $k\in\mathbb{N}$. Moreover,

$$R-\varphi_A^k(R)+\varphi_A^k(I)\leq R+\varphi_A^k(R)\leq (1+M)I$$

and (iii) holds. The implication (iii) $\Rightarrow$ (i) follows from Proposition 2.6 of [29]. Indeed, if (iii) holds, then using [29], we find a sequence $\{T_i\}_{i=1}^n$ of operators such that $[T_1,\dots,T_n]$ is a row contraction and $A_i=YT_iY^{-1}$, where $Y\in B(\mathcal{H})$ is an invertible positive operator and $\sqrt{a}I\leq Y\leq\sqrt{b}I$. Set $\varphi_T(X):=\sum_{i=1}^n T_iXT_i^*$ and notice that $\varphi_T(I)\leq I$ and φ_A is similar to φ_T. On the other hand, we have

$$p(A_1,\dots,A_n)=Yp(T_1,\dots,T_n)Y^{-1}$$

for any polynomial $p(S_1,\dots,S_n)\in\mathcal{A}_n$. Hence, we infer (iv) and, using the noncommutative von Neumann inequality [33], we get

$$\|\Phi\|_{cb}\leq\|Y\|\|Y^{-1}\|\leq\sqrt{\frac{b}{a}}.$$

If we assume that (iv) holds, then using Paulsen's result [26] and [36], we infer that $\{A_i\}_{i=1}^n$ is simultaneously similar to $\{T_i\}_{i=1}^n$. This implies that φ_A is similar to φ_T. The proof is complete. $\square$

6. Numerical invariants for Hilbert modules over free semigroup algebras

The Poisson transforms of Section 2 are used in this section to define certain numerical invariants associated with (not necessarily contractive) Hilbert modules over the free semigroup algebra $\mathbb{C}\mathbb{F}_n^+$. Any Hilbert module $\mathcal{H}$ over $\mathbb{C}\mathbb{F}_n^+$ corresponds to a unique w^*-continuous completely positive map φ on $B(\mathcal{H})$ and therefore to a unique noncommutative cone $C_{\leq}(\varphi)^+$. A notion of $*$-curvature $\operatorname{curv}_*(\varphi,D)$ and Euler characteristic $\chi(\varphi,D)$ are associated with each ordered pair (φ,D), where $D\in C_{\leq}(\varphi)^+$. In this section, we obtain asymptotic formulas and basic properties for both the $*$-curvature and the Euler characteristic associated with (φ,D). In the particular case when $\mathcal{H}$ is a contractive Hilbert modules over $\mathbb{C}\mathbb{F}_n^+$ and $D:=I$, our two variable invariant

$$F(\varphi,I):=(\|\varphi^*(I)\|,\ \operatorname{curv}_*(\varphi,I))$$

is a refinement of the curvature invariant from [39] and [24].

Let $\mathbb{C}\mathbb{F}_n^+$ be the complex free semigroup algebra generated by the free semigroup $\mathbb{F}_n^+$ with generators $g_1,\dots,g_n$ and neutral element e. Any n-tuple $T_1,\dots,T_n$ of bounded operators on a Hilbert space $\mathcal{H}$ gives rise to a Hilbert (left) module over $\mathbb{C}\mathbb{F}_n^+$ in the natural way

$$f\cdot h:=f(T_1,\dots,T_n)h,\quad f\in\mathbb{C}\mathbb{F}_n^+, h\in\mathcal{H}.$$

We associate with the canonical operators $T_1,\dots,T_n$ the completely positive linear map

$$\varphi(X):=\sum_{i=1}^n T_iXT_i^*,\quad X\in B(\mathcal{H}).$$

The *adjoint* of φ is defined by $\varphi^*(X):=\sum_{i=1}^n T_i^*XT_i$. We associate with φ and each positive operator $D\in B(\mathcal{H})$ such that $\varphi(D)\leq D$ a two variable numerical invariant

$$F(\varphi,D):=(\|\varphi^*(I)\|,\ \operatorname{curv}_*(\varphi,D)),\tag{6.1}$$

where the $*$-curvature is defined by

$$\operatorname{curv}_*(\varphi, D) := \lim_{k\to\infty} \frac{\operatorname{trace}[K_{\varphi,D}^*(P_{\leq k}\otimes I)K_{\varphi,D}]}{1+\|\varphi^*(I)\|+\cdots+\|\varphi^*(I)\|^k}. \tag{6.2}$$

Here, $K_{\varphi,D}$ is the Poisson kernel associated with φ and D (see Section 2). Notice that the operator $K_{\varphi,D}^*(P_{\leq k}\otimes I)K_{\varphi,D}$ is the Poisson transform of the orthogonal projection $P_{\leq k} := I - \sum_{|\alpha|=k+1} S_\alpha S_\alpha^*$.

In what follows we show that the limit defining the $*$-curvature exists.

Theorem 6.1. *Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ such that $\varphi(X)=\sum_{i=1}^n T_iXT_i^*$ $(n\in\mathbb{N})$. If $D\in B(\mathcal{H})$ is a positive operator such that $\varphi(D)\leq D$, then*

$$\operatorname{curv}_*(\varphi, D) := \lim_{k\to\infty} \frac{\operatorname{trace}[K_{\varphi,D}^*(P_{\leq k}\otimes I)K_{\varphi,D}]}{1+\|\varphi^*(I)\|+\cdots+\|\varphi^*(I)\|^k} \tag{6.3}$$

exists. Moreover, $\operatorname{curv}_*(\varphi, D)<\infty$ *if and only if* $\operatorname{trace}(D-\varphi(D))<\infty$.

Proof. Due to the properties of the Poisson kernel $K_{\varphi,D}$ of Section 2 (see relation (2.3)), we have

$$\begin{aligned}
K_{\varphi,D}^*(P_{\leq k}\otimes I)K_{\varphi,D} &= K_{\varphi,D}^*(I-\phi_S^{k+1}(I)\otimes I)K_{\varphi,D}\\
&= K_{\varphi,D}^*K_{\varphi,D} - K_{\varphi,D}^*(\phi_S^{k+1}(I)\otimes I)K_{\varphi,D}\\
&= D-\varphi^\infty(D)-\varphi^{k+1}(D)+\varphi^\infty(D)\\
&= D-\varphi^{k+1}(D),
\end{aligned}$$

where $\phi_S(Y):=\sum_{i=1}^n S_iYS_i^*$ and $S_1,\dots,S_n$ are the left creation operators on the full Fock space $F^2(H_n)$. Since the sequence $\{D-\varphi^k(D)\}_{k=1}^\infty$ is increasing, it is clear that $\operatorname{curv}_*(\varphi,D)=\infty$ whenever $\operatorname{trace}(D-\varphi(D))=\infty$. Assume now that $\operatorname{trace}(D-\varphi(D))<\infty$. First we consider the case when $\|\varphi^*(I)\|>1$. Using the definition (6.2), we have

$$\operatorname{curv}_*(\varphi,D) = (\|\varphi^*(I)\|-1)\lim_{k\to\infty}\frac{\operatorname{trace}[D-\varphi^k(D)]}{\|\varphi^*(I)\|^k}.$$

Let us show that this limit exists. If $X\geq 0$ is a trace class operator, then

$$\begin{aligned}
\operatorname{trace}\varphi(X) &= \operatorname{trace}\sum_{i=1}^n X^{1/2}A_i^*A_iX^{1/2} = \operatorname{trace}[X^{1/2}\varphi^*(I)X^{1/2}]\\
&\leq \|\varphi^*(I)\|\operatorname{trace} X.
\end{aligned} \tag{6.4}$$

Since

$$D-\varphi^{k+1}(D) = D-\varphi(D)+\varphi(D-\varphi^k(D)),\ \ k=1,2,\dots, \tag{6.5}$$

we infer that $D-\varphi^{k+1}(D)$ is a trace class operator. From relations (6.4) and (6.5), we obtain

$$\operatorname{trace}[D-\varphi^{k+1}(D)] \leq \|\varphi^*(I)\|\operatorname{trace}[D-\varphi^k(D)]+\operatorname{trace}[D-\varphi(D)]. \tag{6.6}$$

Therefore, setting

$$a_k := \frac{\operatorname{trace}[D-\varphi^k(D)]}{\|\varphi^*(I)\|^k} - \frac{\operatorname{trace}[D-\varphi^{k-1}(D)]}{\|\varphi^*(I)\|^{k-1}},$$

we have

$$a_k \leq \frac{\operatorname{trace}[D-\varphi(D)]}{\|\varphi^*(I)\|^k}, \quad k=1,2,\dots.$$

Notice that $\sum_{k:\, a_k \geq 0} a_k < \infty$. Furthermore, every partial sum of the negative a_k's is greater then or equal to $-\text{trace}\,[D - \varphi(D)]$, so $\sum_{k:\, a_k<0} a_k$ converges as well. Therefore,

$$\lim_{k\to\infty} \frac{\text{trace}\,[D - \varphi^k(D)]}{\|\varphi^*(I)\|^k}$$

exists.

Now, assume that $\|\varphi^*(I)\| \leq 1$. Using relation (6.4), we get

$$0 \leq \text{trace}\,[\varphi^{k+1}(D - \varphi(D))] \leq \|\varphi^*(I)\|\text{trace}\,[\varphi^k(D - \varphi(D))]$$

for any $k = 0, 1, \ldots$. Hence, the sequence $\{\text{trace}\,[\varphi^k(D - \varphi(D))]\}_{k=0}^\infty$ is decreasing and

$$\lim_{k\to\infty} \text{trace}\,[\varphi^k(D - \varphi(D))]$$

exists. If $\|\varphi^*(I)\| = 1$, then, using an elementary classical result and previous computations, we obtain

$$\begin{aligned}\text{curv}_*(\varphi, D) :&= \lim_{k\to\infty} \frac{\text{trace}\,[K_{\varphi,D}^*(P_{\leq k} \otimes I)K_{\varphi,D}]}{n} \\ &= \lim_{k\to\infty} \text{trace}\,[K_{\varphi,D}^*(P_k \otimes I)K_{\varphi,D}] \\ &= \lim_{k\to\infty} \text{trace}\,[\varphi^k(D - \varphi(D))],\end{aligned}$$

where $P_k := P_{\leq k} - P_{\leq k-1}$. Now, assume $\|\varphi^*(I)\| < 1$. According to the definition, we get

$$\text{curv}_*(\varphi, D) = (1 - \|\varphi^*(I)\|) \lim_{k\to\infty} \text{trace}\,[D - \varphi^k(D)].$$

Iterating relation (6.6), we deduce that the sequence $\{\text{trace}\,[D - \varphi^k(D)]\}_{k=1}^\infty$ is bounded. Since $\{D - \varphi^k(D)\}_{k=1}^\infty$ is increasing, we infer that the above limit exists. The proof is complete. $\square$

Corollary 6.2. *If φ and D are as in Theorem 6.1, then*

$$\text{curv}_*(\varphi, D) = \begin{cases} (\|\varphi^*(I)\| - 1) \lim_{k\to\infty} \frac{\text{trace}\,[D - \varphi^k(D)]}{\|\varphi^*(I)\|^k} & \textit{if } \ \|\varphi^*(I)\| > 1, \\ \lim_{k\to\infty} \text{trace}\,[\varphi^k(D - \varphi(D))] & \textit{if } \ \|\varphi^*(I)\| = 1, \\ (1 - \|\varphi^*(I)\|) \lim_{k\to\infty} \text{trace}\,[D - \varphi^k(D)] & \textit{if } \ \|\varphi^*(I)\| < 1. \end{cases} \tag{6.7}$$

Let Λ be a nonempty set of positive numbers $\alpha > 0$ such that

$$\text{trace}\,\varphi(X) \leq \alpha\, \text{trace}\, X$$

for any positive trace class operator $X \in B(\mathcal{H})$. Notice that Theorem 6.1 and Corollary 6.2 remain true (with exactly the same proofs) if we replace $\|\varphi^*(I)\|$ with $\alpha \in \Lambda$. The corresponding curvature is denoted by $\text{curv}_\alpha(\varphi, D)$.

When $d := \inf \Lambda$, the curvature $\text{curv}_d(\varphi, D)$ is called the distinguished curvature associated with (φ, D) and with respect to Λ. Now, using the analogues of Theorem 6.1 and Corollary 6.2 for the curvatures $\text{curv}_\alpha(\varphi, D)$, $\alpha \in \Lambda$, one can easily prove the following. If $\text{curv}_\alpha(\varphi, D) > 0$, then

$$\text{curv}_\alpha(\varphi, D) = \begin{cases} \text{curv}_d(\varphi, D) & \text{if } \ \alpha \geq 1, \\ \frac{1-\alpha}{1-d}\, \text{curv}_d(\varphi, D) & \text{if } \ \alpha < 1. \end{cases}$$

As we will see later in the paper, if $\operatorname{curv}_{\alpha_0}(\varphi, D) = 0$ for some $\alpha_0 \in \Lambda$, then, in general, $\operatorname{curv}_d(\varphi, D) \neq 0$. Therefore, the distinguished curvature $\operatorname{curv}_d(\varphi, D)$ is a refinement of all the other curvatures $\operatorname{curv}_\alpha(\varphi, D)$, $\alpha \in \Lambda$.

Let us consider an important particular case. According to the inequality (6.4), we can take Λ to be the set of all positive numbers $\alpha = \|\sum_{i=1}^m T_i^* T_i\|$, where $(T_1, \dots, T_m)$ is any m-tuple representing the completely positive map φ, i.e.,

$$\varphi(X) = \sum_{i=1}^m T_i X T_i^*, \quad X \in B(\mathcal{H}).$$

Therefore, we can talk about a distinguished curvature associated with (φ, D). All the results of this section concerning the $*$-curvature have analogues (and similar proofs) for the distinguished curvature.

From now on, for the sake of simplicity, we assume that $D - \varphi(D)$ is a finite rank operator. Using formula (6.7), we can prove some properties of the $*$-curvature. We consider only the case when $\|\varphi^*(I)\| > 1$. The other cases can be treated similarly, but we leave this task to the reader.

Theorem 6.3. *Let φ and ψ be w^*-continuous completely positive linear maps on $B(\mathcal{H})$ and $B(\mathcal{H}')$, respectively.*

(i) *If $X \in B(\mathcal{H})$, $Y \in B(\mathcal{H}')$ are positive operators such that $\varphi(X) \leq X$ and $\psi(Y) \leq Y$, then*

$$\operatorname{curv}_*(\varphi \oplus \psi, X \oplus Y) = \begin{cases} \operatorname{curv}_*(\varphi, X) + \operatorname{curv}_*(\psi, Y) & \text{if } \|\varphi^*(I)\| = \|\psi^*(I)\|, \\ \operatorname{curv}_*(\varphi, X) & \text{if } \|\varphi^*(I)\| > \|\psi^*(I)\|, \\ \operatorname{curv}_*(\psi, Y) & \text{if } \|\varphi^*(I)\| < \|\psi^*(I)\|. \end{cases}$$

(ii) *If $D_j \in B(\mathcal{H})$, $j = 1, 2$, are positive operators such that $\varphi(D_j) \leq D_j$, then*

$$\operatorname{curv}_*(\varphi, c_1 D_1 + c_2 D_2) = c_1 \operatorname{curv}_*(\varphi, D_1) + c_2 \operatorname{curv}_*(\varphi, D_2),$$

for any positive constants c_1, c_2.

(iii) *If D is a positive operator such that $\varphi(D) \leq D$, then*

$$\operatorname{curv}_*(\varphi, D) \leq \operatorname{trace}[D - \varphi(D)] \leq \|D - \varphi(D)\| \operatorname{rank}[D - \varphi(D)].$$

Proof. According to Corollary 6.2 and taking into account that

$$\|(\varphi \oplus \psi)^*(I)\| = \max\{\|\varphi^*(I)\|, \|\psi^*(I)\|\},$$

we have

$$\begin{aligned} \operatorname{curv}_*(\varphi \oplus \psi, X \oplus Y) &= \operatorname{curv}_*(\varphi, X) \lim_{k\to\infty} \left(\frac{\|\varphi^*(I)\|}{\|(\varphi \oplus \psi)^*(I)\|} \right)^k \cdot \frac{\|(\varphi \oplus \psi)^*(I)\| - 1}{\|\varphi^*(I)\| - 1} \\ &+ \operatorname{curv}_*(\psi, Y) \lim_{k\to\infty} \left(\frac{\|\psi^*(I)\|}{\|(\varphi \oplus \psi)^*(I)\|} \right)^k \cdot \frac{\|(\varphi \oplus \psi)^*(I)\| - 1}{\|\psi^*(I)\| - 1}. \end{aligned}$$

Hence, (i) follows. To prove (ii), notice that if $c_j \geq 0$ and $\varphi(D_j) \leq D_j$, then

$$\varphi(c_1 D_1 + c_2 D_2) \leq c_1 D_1 + c_2 D_2.$$

Taking into account Corollary 6.2 and the linearity of the trace, we complete the proof of (ii). Now, using the inequality (6.6), we deduce

$$\begin{aligned}\frac{\operatorname{trace}[D-\varphi^{k+1}(D)]}{\|\varphi^*(I)\|^{k+1}} &\leq \sum_{m=1}^{k+1} \frac{\operatorname{trace}[D-\varphi(D)]}{\|\varphi^*(I)\|^m} \\ &= \frac{\operatorname{trace}[D-\varphi(D)]}{\|\varphi^*(I)\|^{k+1}} \cdot \frac{\|\varphi^*(I)\|^{k+1}-1}{\|\varphi^*(I)\|-1}.\end{aligned}$$

Hence, and using relation (6.3), we have

$$\begin{aligned}\operatorname{curv}_*(\varphi, D) &\leq \operatorname{trace}[D-\varphi(D)] \\ &\leq \|D-\varphi(D)\| \operatorname{rank}[D-\varphi(D)],\end{aligned}$$

and (iii) follows. The proof is complete. □

Corollary 6.4. *Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$, and let $D \in B(\mathcal{H})$, $D \geq 0$, be such that $\varphi(D) \leq D$. If $D = R + Q$ is the canonical decomposition of D with respect to φ, then*

$$\operatorname{curv}_*(\varphi, D) = \operatorname{curv}_*(\varphi, Q),$$

where Q is the pure part of D.

Using a result from [39], we can prove the following characterization of Hilbert modules isomorphic to finite rank free Hilbert modules.

Proposition 6.5. *A pure Hilbert module $\mathcal{H}$ over $\mathbb{C}\mathbb{F}_n^+$ is isomorphic to a finite rank free Hilbert module $F^2(H_n) \otimes \mathcal{K}$, where $\mathcal{K}$ is a Hilbert space, if and only if $\varphi(I) \leq I$ and*

$$F(\varphi, I) = (n, \operatorname{rank}\mathcal{H}), \tag{6.8}$$

where φ is the completely positive linear map associated with the Hilbert module $\mathcal{H}$. Moreover, in this case, $\dim \mathcal{K} = \operatorname{rank}\mathcal{H}$.

Proof. Let $T_1, \ldots, T_n$ be the canonical operators associated with $\mathcal{H}$. Assume that $\mathcal{H}$ is isomorphic to a finite rank free Hilbert module $F^2(H_n) \otimes \mathcal{K}$, i.e., $\mathcal{K}$ is a finite dimensional Hilbert space and there is a unitary operator $U : \mathcal{H} \to F^2(H_n) \otimes \mathcal{K}$ such that

$$T_i = U^*(S_i \otimes I_{\mathcal{K}})U, \quad i = 1, \ldots, n.$$

A simple calculation shows that:

(i) $\operatorname{rank}\mathcal{H} := \dim \overline{(I - \sum_{i=1}^n T_i T_i^*)\mathcal{H}} = \dim \mathcal{K}$;
(ii) $\|\varphi^*(I)\| = n$;
(iii) $\operatorname{curv}_*(\varphi, I) = \dim \mathcal{K}$.

The latter equality is a consequence of Corollary 6.2. Therefore, relation (6.8) is satisfied.

Conversely, assume that $\mathcal{H}$ is a pure Hilbert module over $\mathbb{C}\mathbb{F}_n^+$ such that $\varphi(I) \leq I$ and relation (6.8) holds. Then $\|\varphi^*(I)\| = n$ and $\operatorname{curv}_*(\varphi, I) = \operatorname{rank}\mathcal{H}$. Hence, we obtain $\operatorname{curv}(\mathcal{H}) = \operatorname{rank}\mathcal{H}$, where

$$\operatorname{curv}(\mathcal{H}) = \begin{cases} (n-1) \lim\limits_{k\to\infty} \frac{\operatorname{trace}[I-\varphi^k(I)]}{n^k} & \text{if } n \geq 2, \\ \lim\limits_{k\to\infty} \operatorname{trace}[\varphi^k(I-\varphi(I))] & \text{if } n = 1, \end{cases} \tag{6.9}$$

is the curvature invariant introduced in [39]. Now, using Theorem 3.4 from [39], we infer that $\mathcal{H}$ is isomorphic to a finite rank free Hilbert module $F^2(H_n)\otimes\mathcal{K}$, where $\mathcal{K}$ is a Hilbert space with $\dim\mathcal{K}=\operatorname{rank}\mathcal{H}$. The proof is complete. □

Let $\mathcal{H}$ be a contractive Hilbert module over $\mathbb{CF}_n^+$ and let φ be the completely positive linear map associated with $\mathcal{H}$. What is the connection between the invariant $F(\varphi, I)$ and the curvature invariant $\operatorname{curv}(\mathcal{H})$? It is clear that if $\|\varphi^*(I)\|=n$, then $F(\varphi,I)=(n,\operatorname{curv}(\mathcal{H}))$. On the other hand, we can prove the following result.

Lemma 6.6. *Let $\mathcal{H}$ be a finite rank contractive Hilbert module over $\mathbb{CF}_n^+$ and let φ be the completely positive linear map associated with $\mathcal{H}$. If $0<\|\varphi^*(I)\|<n$, then* $\operatorname{curv}(\mathcal{H})=0$.

Proof. First, consider the case $n\geq 2$. Assume $\operatorname{curv}(\mathcal{H})>0$ and $\|\varphi^*(I)\|>1$. Since $\mathcal{H}$ be a finite rank contractive Hilbert module, using Theorem 6.3, we infer that

$$\operatorname{curv}_*(\varphi,I)\leq \operatorname{rank}\,[I-\varphi(I)]<\infty.$$

Taking into account relations (6.7) and (6.9), we have

$$\frac{\|\varphi^*(I)\|-1)}{n-1}\lim_{k\to\infty}\frac{\operatorname{trace}\,[I-\varphi^k(I)]}{\|\varphi^*(I)\|^k}\cdot\left[\frac{\operatorname{trace}\,[I-\varphi^k(I)]}{n^k}\right]^{-1}=\frac{\operatorname{curv}_*(\varphi,I)}{\operatorname{curv}(\mathcal{H})}<\infty.$$

Hence, we infer that $\lim\limits_{k\to\infty}\frac{n^k}{\|\varphi^*(I)\|^k}<\infty$, which a contradiction. Hence, $\operatorname{curv}(\mathcal{H})=0$. Similarly, one can show that the same conclusion holds if $0<\|\varphi^*(I)\|\leq 1$.

Now, consider the case $n=1$. According to [39], we have

$$\operatorname{curv}(\mathcal{H})=\lim_{k\to\infty}\frac{\operatorname{trace}\,[I-\varphi^k(I)]}{k}$$

As above, taking again into account relation (6.7) (the case $\|\varphi^*(I)\|<1$) , one can easily show that $\operatorname{curv}(\mathcal{H})=0$. □

Therefore, the curvature invariant $\operatorname{curv}(\mathcal{H})$ does not distinguish among the Hilbert modules over $\mathbb{CF}_n^+$ with $0<\|\varphi^*(I)\|<n$. However, in this case, our $*$-curvature $\operatorname{curv}_*(\varphi,I)$ in not zero in general. More precisely, we can prove the following.

Proposition 6.7. *If $m=2,3,\ldots,n-1$, and $t\in(0,1]$, then there exists a finite rank contractive Hilbert module $\mathcal{H}$ over $\mathbb{CF}_n^+$, such that $\|\varphi^*(I)\|=m$ and $\operatorname{curv}_*(\varphi,I)=t$, i.e.,*

$$F(\varphi,I)=(m,t).$$

Proof. According to Theorem 3.8 from [39], there is a finite rank contractive Hilbert module $\mathcal{H}$ over $\mathbb{CF}_m^+$ such that $\operatorname{curv}(\mathcal{H})=t$. Let $T:=[T_1,\ldots,T_m]$ be the row contraction associated with $\mathcal{H}$ and let φ_T be the corresponding completely positive map. Since $t>0$, Lemma 6.6 implies $\|\varphi^*(I)\|=m$. Let $\mathcal{H}'$ be the finite rank contractive Hilbert module over $\mathbb{CF}_n^+$ defined by the row contraction

$$[T_1,\ldots,T_{m-1},\frac{1}{\sqrt{n-m+1}}T_m,\ldots,\frac{1}{\sqrt{n-m+1}}T_m],$$

and let φ be the associated completely positive map. Notice that $\varphi(X)=\varphi_T(X)$, $X\in B(\mathcal{H})$, and $\varphi^*(I)=\varphi_T^*(I)$. Hence, we have $\|\varphi^*(I)\|=m$. Taking into account relations (6.7) and (6.9),

we infer that

$$\begin{aligned}\operatorname{curv}_*(\varphi, I) &= (m-1)\lim_{k\to\infty}\frac{\operatorname{trace}[I-\varphi_T^k(I)]}{m^k}\\ &= \operatorname{curv}(\mathcal{H}) = t.\end{aligned}$$

Summing up, we have $F(\varphi, I) = (m, t)$, which completes the proof. □

This result clearly shows that, in the particular case of finite rank contractive Hilbert modules over $\mathbb{C}\mathbb{F}_n^+$, our invariant $F(\varphi, I)$ is a refinement of $\operatorname{curv}(\mathcal{H})$.

Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}\mathbb{F}_n^+$, φ be its associated completely positive map, and let D be a positive operator such that $\varphi(D) \leq D$. For each $k = 0, 1, \ldots,$ define

$$M_k(\varphi, D) := \operatorname{span}\left\{p\cdot\xi :\ p\in\mathbb{C}\mathbb{F}_n^+,\ \deg(p)\leq k,\ \xi\in[D-\varphi(D)]^{1/2}\mathcal{H}\right\}.$$

We define the Euler characteristic associated with φ and D by setting

$$\chi(\varphi, D) := \lim_{k\to\infty}\frac{\dim M_k(\varphi, D)}{1+n+\cdots+n^k}. \tag{6.10}$$

In what follows we show that the limit (6.10) exists (finite or infinite).

Theorem 6.8. *Let φ be a w^*-continuous completely positive linear map on $B(\mathcal{H})$ such that $\varphi(X) = \sum_{i=1}^n T_i X T_i^*$ $(n \geq 2)$, and let D be a positive operator such that $\varphi(D) \leq D$. Then the Euler characteristic $\chi(\varphi, D)$ exists and*

$$\begin{aligned}\chi(\varphi, D) &= \lim_{k\to\infty}\frac{\operatorname{rank}[K_{\varphi,D}^*(P_{\leq k}\otimes I)K_{\varphi,D}]}{1+n+\cdots+n^k}\\ &= (n-1)\lim_{k\to\infty}\frac{\operatorname{rank}[D-\varphi^k(D)]}{n^k}.\end{aligned}$$

Moreover, the Euler characteristic $\chi(\varphi, D) < \infty$ if and only if $\operatorname{rank}[D-\varphi(D)] < \infty$.

Proof. Notice that if $\operatorname{rank}[D-\varphi(D)] = \infty$, then the inclusion $M_0(\varphi, D) \subset M_k(\varphi, D)$ implies $\chi(\varphi, D) = \infty$. Now, assume that $\operatorname{rank}[D-\varphi(D)] < \infty$. Notice that $M_k(\varphi, D)$ is equal to the range of $K_{\varphi,D}^*(P_{\leq k}\otimes I)$. Since the latter operator has finite rank, we have

$$\dim M_k(\varphi, D) = \operatorname{rank}[K_{\varphi,D}^*(P_{\leq k}\otimes I)K_{\varphi,D}] = \operatorname{rank}[D-\varphi^{k+1}(D)].$$

For each $k \geq 1$, we have

$$M_k(\varphi, D) = M_0(\varphi, D) + T_1 M_{k-1}(\varphi, D) + \cdots + T_n M_{k-1}(\varphi, D).$$

Hence, we infer

$$\dim M_k(\varphi, D) \leq n\dim M_{k-1}(\varphi, D) + \operatorname{rank}[D-\varphi(D)], \quad k = 1, 2, \ldots.$$

The rest of the proof is similar to that of Theorem 4.1 from [39]. We shall omit it. □

We should mention that in the particular case when $\varphi(I) \leq I$ and $D = I$, the result of Theorem 6.8 was obtain in [39].

Similarly to the proof of Theorem 6.3, one can prove the following result.

Proposition 6.9. *For each $j = 1, 2$, let $\mathcal{H}_j$ be a Hilbert module over $\mathbb{C}\mathbb{F}_n^+$, and φ_j be its associated completely positive linear map. Let D_j be positive operators such that $\varphi_j(D) \leq D_j$. Then we have:*

(i) $\chi(\varphi_1 \oplus \varphi_2, D_1 \oplus D_2) = \chi(\varphi_1, D_1) + \chi(\varphi_2, D_2)$.
(ii) *If* $D_1 \leq I$ *and* $\|\varphi_1^*(I)\| = n$, *then* $\operatorname{curv}_*(\varphi_1, D_1) \leq \chi(\varphi_1, D_1)$.

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